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Expo. Math. 31 (2013) 40–72

EXPOSITIONES MATHEMATICAE

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Mixed multiplier ideals and the irregularity of abelian coverings of smooth projective surfaces

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Received 7 November 2011; received in revised form 30 March 2012

Abstract

A formula for the irregularity of abelian coverings of smooth projective surfaces is established. Explicit computations are performed and some applications are presented.

MSC 2010: primary 14E20; secondary 14H20; 14Jxx

Keywords: Algebraic surface; Abelian covering; Mixed multiplier ideal

1. Introduction

The aim of this paper is to establish a formula for the irregularity of abelian coverings of smooth projective surfaces using, on the one hand, the theory of abelian coverings developed in [22], and on the other hand, the theory of mixed multiplier ideals defined in [12, Generalization 9.2.8]. Such a formula was first developed in Zariski's paper [27] for cyclic coverings of the projective plane: to a plane curve C transverse to the line at infinity H_0 and with only nodes and cusps as singularities, a family of cyclic coverings is associated. A covering from the family is ramified over C and possibly H_0 and is determined by the degree n of the covering—over the affine plane its equation is $z^n = f(x, y)$, with $f = 0$ an affine equation of C. If d is the degree of the branch curve C, Z is the set of its cusps, and S_n is a desingularization, then the irregularity of S_n equals

$$h^1\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\left(-3 + \frac{5d}{6}\right) \otimes \mathcal{I}_Z\right),$$

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2012.08.005

whenever 6 divides d and n, and equals 0 otherwise. The rational number $5/6$ is the first jumping number of a cusp—its log canonical threshold—and the ideal sheaf $\mathcal{I}_Z$, in this context, is the corresponding multiplier ideal of C,

$$\mathcal{I}_Z = \mathcal{J}\left(\frac{5}{6}\cdot C\right).$$

Removing the restriction on the singularities, Zariski's formula for cyclic coverings of the plane becomes

$$q(S_n) = \sum_{\substack{\xi \text{ jumping number of } C \\ \xi \in 1/(n \wedge \deg C)\mathbb{Z},\ 0<\xi<1}} h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3+\xi \deg C)\otimes \mathcal{J}(\xi \cdot C)). \tag{1}$$

Under different disguises, (1) was developed in [13,3,15,14,19,25,20]. One of its main features is that the right hand side is a quasi-constant function of n, the degree of the covering.

When the curve C is no longer transverse to the line at infinity, the irregularity might behave differently. In Example 5.4, the cyclic covering is ramified over the union of two smooth conics with common tangents at two points; the irregularity is a degree 1 quasi-polynomial of n when the line at infinity is the line through the two tangency points. To understand this difference, we extend formula (1) to a non transverse configuration by considering the setting of abelian coverings. In doing so, we encounter two questions: (1) what are the abelian coverings we intend to study? and (2) what is the "object" that holds the role of a jumping number when the role played by the multiplier ideals will be taken on by the mixed multiplier ideals?

To answer the latter, we see that if $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ are non-zero ideal sheaves on a smooth surface, the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{x^1}\cdots\mathfrak{a}_t^{x^t})$ varies with the rational vector $\boldsymbol{x} = (x^1, \ldots, x^t) \in \mathbb{R}^t_+$. Propositions 2.2 and 2.7 assert that there is a set of hyperplanes called *jumping walls*, satisfying the following properties.

- If the mixed multiplier ideal jumps, then the vector $\boldsymbol{x}$ crosses a jumping wall. Consequently, the fibres of the map $\boldsymbol{x} \mapsto \mathcal{J}(\mathfrak{a}_1^{x^1}\cdots\mathfrak{a}_t^{x^t})$ are finite unions of rational convex polytopes cut out by the jumping walls.
- The jumping walls are determined by some of the *jumping numbers* of the *simple complete relevant ideals* (see Definition 2.6) associated to the ideals $\mathfrak{a}_i$.

These results together with Pardini's characterization of standard abelian coverings enable us, in Theorem 4.2, to generalize formula (1) to standard abelian coverings of smooth projective surfaces. Thus, we are led to answer the former question.

An abelian covering induces a partition of the branch curve (see Remark 3.2), *i.e.* a decomposition $C = \sum_i C_i$, with C_i reduced but not necessarily irreducible. We assume that C is endowed with an H-partition:

(∗) there exists an ample divisor H such that for each i,
C_i is linearly equivalent to a multiple of H.

With this hypothesis, the irregularity is expressed in Theorem 4.2 as a linear combination of superabundances of linear systems defined in terms of some mixed multiplier ideals associated to the partition $(C_1, \ldots, C_t)$. There exists a natural map from the characters of the Galois group of the covering to the first orthant appearing in the definition of the mixed multiplier ideals; the coefficient of each superabundance represents the number of characters that lie in the distinguished edge (see Definition 3.9) given by the intersection of the jumping walls associated to the corresponding mixed multiplier ideals. We refer the reader to Theorem 4.2 for the precise formula that generalizes Vaquié's formula in [25] for cyclic coverings of projective surfaces, and ours for the irregularity of cyclic coverings of the plane in [20].

Without the hypothesis $(*)$ on the curves C_i, the formula for the irregularity in Theorem 4.2 is no longer available. Nevertheless, a less explicit formula holds true; see [17,1]. We reprove it in Theorem 4.1.

Some applications of Theorem 4.2 are presented in Section 5: the discussion of the general cyclic coverings of the plane; Hironaka's result from [6] concerning the asymptotic behaviour of the irregularity of the abelian coverings; the computation of the irregularity of the Hirzebruch surfaces constructed in [8]—abelian coverings of the plane branched along configurations of lines, *i.e.* line arrangements; an example appearing in [10] that uses the general form of the description of an abelian covering.

Generalizations of formula (1) to abelian coverings, using different approaches, appear in [14,15,17], and [1]. Libgober establishes in [15, Section 3.1] a formula for the irregularity of abelian coverings of the plane and extends it to abelian coverings of $\mathbb{P}^d$ in [17], his technique being based on mixed Hodge structures. His formula—a sum of superabundances of linear systems expressed in terms of quasiadjunction ideals with coefficients given by quasiadjunction polytopes—and ours bear clear resemblances. We refer to [16] for the relation between the quasiadjunction ideals and the multiplier ideals. In [1, Corollary 1.13], dealing with abelian coverings of a smooth projective variety X of dimension d, Budur obtains the Hodge numbers $h^{0,q}, 0 \leq q \leq d$, as sums of "superabundances". Moreover, in [1, Theorem 1.3] and in the proof of [1, Theorem 1.8], the terms of those sums are grouped together, the coefficients being given by the number of certain rational points inside convex polytopes. Budur's work is constructed around the study of the space of unitary local systems of rank one on the complement of an arbitrary divisor $D \subset X$. In [1, Example 6.6.(b)], he notices that in applications, the method for the computations inherent to his formula is different from Libgober's. It is also different from ours. When applying Theorem 4.2 to compute the irregularity of an abelian covering, we rely on the technical result in Proposition 2.7, specific to the dimension 2. It allows us to considerably restrict the search for the jumping walls. Afterwards, the jumping walls and the partition of the branch curve determine the distinguished edges, and each distinguished edge determines a unique linear system—a set of mixed multiplier ideals and a power of the ample divisor H appearing in $(*)$.

Notation

All varieties are assumed to be defined over $\mathbb{C}$ and standard symbols and notation in algebraic geometry will be used.

If x is a real number, then $\lfloor x \rfloor$ is the integer part of x, $\lceil x \rceil$ is the smallest integer not smaller than x, and $\langle x \rangle$ is the fractional part of x, *i.e.* $x = \lfloor x \rfloor + \langle x \rangle$. If a denotes a class in $\mathbb{Z}/m\mathbb{Z}$, then $a^{\bullet}$ is the smallest non-negative integer in the equivalence class a.

2. Mixed multiplier ideals

In this section we define and characterize the jumping walls associated to mixed multiplier ideals. We start by briefly recalling the notions of multiplier ideals and mixed multiplier ideals for ideal sheaves on a smooth surface following [12].

Mixed multiplier ideals and jumping walls

Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X and let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. If ξ is a positive rational number, then the multiplier ideal associated to ξ and $\mathfrak{a}$ is defined as

$$\mathcal{J}(\mathfrak{a}^{\xi}) = \mu_* \mathcal{O}_Y(K_\mu - \lfloor \xi F \rfloor).$$

Now, for the analogous notion for several ideals, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals and $\mu : Y \to X$ a common log resolution of the ideals $\mathfrak{a}_i$ with $\mathfrak{a}_i \cdot \mathcal{O}_X = \mathcal{O}_Y(-F_i)$ and $\sum_i F_i + \text{except}(\mu)$ having simple normal crossing support. If $\xi_1, \ldots, \xi_t$ are positive rational numbers, then the *mixed multiplier ideal* associated to the ξ_i and the $\mathfrak{a}_i$ is

$$\mathcal{J}(\mathfrak{a}_1^{\xi_1} \cdots \mathfrak{a}_t^{\xi_t}) = \mu_* \mathcal{O}_Y(K_\mu - \lfloor \xi_1 F_1 + \cdots + \xi_t F_t \rfloor).$$

Definition–Lemma. *(see* [12, *Lemma 9.3.21*]*). Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X and let $P \in X$ be a fixed point in the support of $\mathfrak{a}$. Then there is an increasing sequence of positive rational numbers $\xi_j = \xi_j(\mathfrak{a}, P)$ such that for every $\xi \in [\xi_j, \xi_{j+1})$,*

$$\mathcal{J}(\mathfrak{a}^{\xi_j}) = \mathcal{J}(\mathfrak{a}^{\xi}) \supset \mathcal{J}(\mathfrak{a}^{\xi_{j+1}}).$$

The rational numbers ξ_j are called the jumping numbers of the ideal sheaf $\mathfrak{a}$ at P.

The multiplier ideals and the jumping numbers are defined similarly in the context of effective $\mathbb{Q}$-divisors. By [12, Proposition 9.2.28], if C is a general element of the ideal sheaf $\mathfrak{a}$ and ξ is a positive rational less than 1, then $\mathcal{J}(\xi \cdot C) = \mathcal{J}(\mathfrak{a}^{\xi})$. Moreover, for any integer divisor C through a point P, the jumping numbers of C at P are periodic and determined by the ones lying in the unit interval $[0, 1)$. Similarly, the jumping numbers of an ideal sheaf $\mathfrak{a}$ at P are periodic and determined by the ones lying in the interval $[0, 2]$. We refer the reader to [12, Example 9.3.24] for more ample details.

For the remainder of this section we consider $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. We want to study the behaviour of the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ as $\boldsymbol{x} = (x^1, \ldots, x^t)$ varies in the first orthant. If $\mu : Y \to X$ is a log resolution defined as before, we shall denote by E_α the irreducible components of the exceptional divisors seen on Y. There exist effective divisors B_α on Y such that (B_α) is the dual basis to $(-E_\alpha)$ of the lattice $\Lambda_\mu = \bigoplus_\alpha \mathbb{Z}E_\alpha$ with respect to the intersection form on Y. The basis (B_α) is called the *branch basis* of the resolution.

Next we want to define the notion of relevant divisors. We follow [23], but see also [4].

Definition 2.1. Let $\mathfrak{a} \subset \mathfrak{m}_P$. An irreducible component E_ρ in a log resolution μ of $\mathfrak{a}$ is called a *relevant divisor* of $\mathfrak{a}$ at P if

$$E_\rho \cdot \mathring{E}_\rho \geq \min(3, 1 - E_\rho^2), \tag{2}$$

where $\mathring{E}_\rho = (\mu^* C)_{red} - E_\rho$ with C the curve defined by a general element[1] of $\mathfrak{a}$. The index ρ will be referred to as a *relevant position*.

Remark. Note that the difference with respect to the notion introduced in [23] comes from the fact that we consider jumping numbers associated to ideal sheaves. In terms of Enriques diagrams, a relevant position for which $E_\rho^2 = -1$ corresponds to an arrowhead vertex of the augmented Enriques tree of C at P. For example, for the ideal $\mathfrak{m}_P^2$, the exceptional divisor of the minimal log resolution becomes a relevant divisor.

The set of relevant positions of $\mathfrak{a}$ at P will be denoted by $\mathfrak{R} = \mathfrak{R}_P(\mathfrak{a})$. The following proposition stresses the importance of the relevant divisors, or positions, in the computation of mixed multiplier ideals. It will further lead us to the notion of jumping walls associated to the ideal sheaf $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_t$ at P.

Proposition 2.2. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_t$ and $\mathfrak{R}$ the set of relevant positions of $\mathfrak{a}$ at P. If x^i are positive rational numbers, then*

$$\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t}) = \mu_* \mathcal{O}_Y \left(K_\mu - \sum_{\rho \in \mathfrak{R}} \left\lfloor \sum_i x^i e_i^\rho \right\rfloor E_\rho \right),$$

where for every i, $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_\alpha e_i^\alpha E_\alpha)$.

Proof. Consider $\boldsymbol{y} = c\boldsymbol{x}$ with $c \in [0, 1]$. If $c = 1$ then $\boldsymbol{y} = \boldsymbol{x}$ and as c decreases, the coefficients $\lfloor \sum_i y^i e_i^\alpha \rfloor$ decrease by discrete jumps. More precisely, there is a finite sequence of rationals $0 = c_{g+1} < c_g < c_{g-1} < \cdots < c_1 < c_0 = 1$ with the following properties holding for any $0 \leq l \leq g$:

(1) for any $c \in [c_{l+1}, c_l)$, and for any $\alpha \notin \mathfrak{R}$, $\lfloor c_{l+1} \sum_i x^i e_i^\alpha \rfloor = \lfloor c \sum_i x^i e_i^\alpha \rfloor$;
(2) there exists $\mathfrak{B}(l)$ disjoint from $\mathfrak{R}$ such that for any $\beta \in \mathfrak{B}(l)$,

$$\left\lfloor c_{l+1} \sum_i x^i e_i^\beta \right\rfloor = \left\lfloor c_l \sum_i x^i e_i^\beta \right\rfloor - 1 = c_l \sum_i x^i e_i^\beta - 1;$$

(3) for any $\alpha \notin \mathfrak{B}(l) \cup \mathfrak{R}$, $\lfloor c_{l+1} \sum_i x^i e_i^\alpha \rfloor = \lfloor c_l \sum_i x^i e_i^\alpha \rfloor$.

Set

$$\Delta_l = -\sum_{\alpha \notin \mathfrak{R}} \left\lfloor c_l \sum_i x^i e_i^\alpha \right\rfloor E_\alpha - \sum_{\rho \in \mathfrak{R}} \left\lfloor \sum_i x^i e_i^\rho \right\rfloor E_\rho.$$

To end the proof, it is sufficient to show that $\mu_* \mathcal{O}_Y(K_\mu + \Delta_{l+1}) = \mu_* \mathcal{O}_Y(K_\mu + \Delta_l)$ for any $0 \leq l < g$. Set $\Gamma = \sum_{\beta \in \mathfrak{B}(l)} E_\beta$. We have the following.

[1] An element $f \in \mathfrak{a}$ is said to be a general element of the ideal $\mathfrak{a}$, if f is a general $\mathbb{C}$-linear combination of a set of generators of $\mathfrak{a}$ (see [12, Definition 9.2.27]). Explicitly, for each irreducible component E_α, the function $C \mapsto E_\alpha \cdot ((\mu^* C)_{red} - E_\alpha)$ gets its minimum on a Zariski open subset of the corresponding projective space. Hence being general for C means belonging to the intersection of these open subsets.

Claim. *For any $\Gamma' \subset \Gamma$ and $E_\gamma \subset \Gamma'$ an irreducible component,*

$$\mu_* \mathcal{O}_Y(K_\mu + \Delta_i + \Gamma' - E_\gamma) = \mu_* \mathcal{O}_Y(K_\mu + \Delta_i + \Gamma').$$

We justify the claim only when x^i are less than 1. The general case is similar, but one needs to consider the general form of [12, Proposition 9.2.28]. Let $C_1, \ldots, C_t$ be the curves defined by general elements in $\mathfrak{a}_i$. Using (1) and (2) above we have

$$\begin{aligned} -\Delta_l \cdot E_\gamma &\geq \sum_\alpha \left\lfloor c_l \sum_i x^i e_i^\alpha \right\rfloor E_\alpha \cdot E_\gamma \\ &> \sum_{\beta \in \mathfrak{B}(l)} c_l \sum_i x^i e_i^\beta \, E_\beta \cdot E_\gamma + \sum_{\alpha \notin \mathfrak{B}(l)} \left(c_l \sum_i x^i e_i^\alpha - 1 \right) E_\alpha \cdot E_\gamma \\ &\quad + \sum_i \left(c_l x^i - 1 \right) \widetilde{C}_i \cdot E_\gamma \\ &= c_l \sum_i x^i \mu^* C_i \cdot E_\gamma - ((\mu^* C)_{red} - \Gamma) \cdot E_\gamma. \end{aligned}$$

Hence

$$\begin{aligned} (\Delta_l + \Gamma' - E_\gamma) \cdot E_\gamma &< ((\mu^* C)_{red} - \Gamma) \cdot E_\gamma + (\Gamma' - E_\gamma) \cdot E_\gamma \\ &\leq E_0^\gamma \cdot E_\gamma \leq 2 \end{aligned} \tag{3}$$

since $\gamma \notin \mathfrak{R}_P$. Now, tensoring the short exact sequence of E_γ in Y with $\mathcal{O}_Y(K_\mu + \Delta_l + \Gamma')$ and pushing it down to X, we get the exact sequence

$$\begin{aligned} 0 &\to \mu_* \mathcal{O}_Y(K_\mu + \Delta_l + \Gamma' - E_\gamma) \to \mu_* \mathcal{O}_Y(K_\mu + \Delta_l + \Gamma') \\ &\to H^0(E_\gamma, K_{E_\gamma} + (\Delta_l + \Gamma' - E_\gamma)|_{E_\gamma}). \end{aligned}$$

By (3), the last term vanishes justifying the claim.

From the properties (2) and (3), $\Delta_{l+1} = \Delta_l + \Gamma$. By repeatedly using the claim, we obtain the result. $\square$

Next, we want to define the *jumping walls* associated to the mixed multiplier ideals $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ when $\boldsymbol{x} = (x^1, \ldots, x^t)$ varies in the first orthant. The idea is that by the previous result, such a mixed multiplier ideal varies only when the point $\boldsymbol{x}$ crosses certain hyperplanes defined by equations corresponding to relevant positions. The defining equation of such a hyperplane is of the form

$$\sum_{i=1}^t x^i e_i^\rho = r, \tag{4}$$

with $\rho \in \mathfrak{R}$ and r a positive integer.

Definitions 2.3. A *relevant value* associated to the relevant position $\rho \in \mathfrak{R}$ of the ideal $\mathfrak{a}_1 \cdots \mathfrak{a}_t$ is a positive integer r such that there exist a point $\boldsymbol{y}$ in the hyperplane $H : \sum_{i=1}^t x^i e_i^\rho = r$ and a neighbourhood V of $\boldsymbol{y}$ with the property that the mixed multiplier

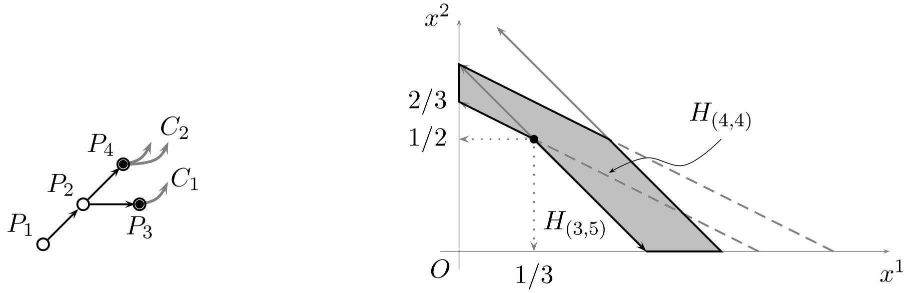

Fig. 1. Log resolution and jumping walls for $\mathfrak{a}_1\mathfrak{a}_2$.

ideal $\mathcal{J}(\mathfrak{a}_1^{x^1}\cdots\mathfrak{a}_t^{x^t})$—corresponding to $\boldsymbol{x} \in V$—changes if and only if $\boldsymbol{x}$ crosses H. The pair (ρ, r) is called a *relevant pair* and the hyperplane H a *jumping wall*.

When we speak of a relevant value, we mean a positive integer which is the relevant value associated to a certain relevant position. Of course, such a value might be associated to many relevant positions, but the position we refer to will be clearly identified in the context.

Remark 2.4. If $\mathfrak{a}$ is a simple complete ideal, the relevant values associated to the relevant position ρ are the integers ξe_ρ, where e_ρ is the coefficient of the irreducible component E_ρ in the minimal log resolution of $\mathfrak{a}$ and ξ runs over all the jumping numbers contributed by ρ. We refer the reader to [11,21] for a formula producing all these jumping numbers.

Example 2.5. Let $\mathfrak{a}_1 = (u^3, v^2)$ and $\mathfrak{a}_2 = (u^6, v^2)$ be ideals in $\mathbb{C}[u, v]$. Let E_1, E_2 and E_3 be the exceptional divisors necessary for the minimal log resolution of $\mathfrak{a}_1$ and let E_4 be the supplementary exceptional divisor necessary for finishing the minimal log resolution of $\mathfrak{a}_2$. Clearly, if C_i are general elements in each $\mathfrak{a}_i$, then $\mu^*(C_1 + C_2) = \widetilde{C}_1 + \widetilde{C}_2 + 4E_1 + 7E_2 + 12E_3 + 9E_4$. The divisors E_3 and E_4 are the only relevant divisors. Then 5 and 7 are the first relevant values associated to the relevant divisor E_3 with the jumping walls $H_{(3,r)} : 6x^1 + 6x^2 = r, r = 5, 7$. Moreover, 4 and 5 are the first relevant values associated to E_4 with the jumping walls $H_{(4,r)} : 3x^1 + 6x^2 = r$, $r = 4, 5$.

The point $\boldsymbol{y}$ from the definition of the jumping wall $H_{(4,4)}$ can be any point on $H_{(4,4)} \cap \mathbb{R}^2_+$ with $y^1 < 1/3$. The other points in the intersection do not satisfy the property in the definition of the relevant value. If $y^1 > 1/3$ then on a sufficiently small neighbourhood of $\boldsymbol{y}$, the mixed multiplier ideal $\mathcal{J}(x^1C_1 + x^2C_2)$ equals the maximal ideal (u, v). If $\boldsymbol{y} = (1/3, 2/3)$ then the multiplier ideal also changes when it crosses the wall $H_{(3,5)}$: $6x^1 + 6x^2 = 5$. In Fig. 1, if $\boldsymbol{x}$ lies in the open shaded polygon, then the mixed multiplier ideal equals the maximal ideal.

Candidates for relevant values

For practical reasons, we need to determine a relatively small set of candidates for the relevant values associated to a relevant position ρ.

Definition 2.6. Let ρ be a relevant position for the ideal $\mathfrak{a}$ and μ a log resolution. The relevant ideal associated to $\mathfrak{a}$ and ρ is the simple complete ideal $\mu_*\mathcal{O}_Y(-B_\rho)$, where B_ρ is the ρ element in the branch basis of the resolution.

Proposition 2.7. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_t$ with (B_α) the branch basis of the resolution. Then the set of relevant values associated to the relevant position ρ is contained in the set of relevant values associated to ρ of the relevant ideal $\mu_*\mathcal{O}_Y(-B_\rho)$.*

Proof. It is sufficient to consider the case $t \geq 2$. Let ρ_0 be a relevant position and r a relevant value with $H : \sum_{i=1}^t x^i e_i^{\rho_0} = r$ the corresponding hyperplane. The point $\boldsymbol{y}$ may be chosen such that H is the only jumping hyperplane containing it. It is here that we need $t \geq 2$. Using Proposition 2.2, since

$$\begin{aligned}\mu_*\mathcal{O}_Y\left(K_\mu - \left\lfloor \sum_i y^i F_i \right\rfloor\right) &\subset \mu_*\mathcal{O}_Y\left(K_\mu - \left\lfloor \sum_i y^i F_i \right\rfloor + E_{\rho_0}\right) \\ &= \mu_*\mathcal{O}_Y\left(K_\mu - \left\lfloor \sum_i (1-\varepsilon) y^i F_i \right\rfloor\right)\end{aligned}$$

with $0 < \varepsilon < 1$, we get that

$$\mu_*\mathcal{O}_Y\left(K_\mu - \sum_{\rho\in\mathfrak{R}} r^\rho E_\rho\right) \subset \mu_*\mathcal{O}_Y\left(K_\mu - \sum_{\rho\neq\rho_0} r^\rho E_\rho - (r-1)E_{\rho_0}\right).$$

Setting $K_\mu = \sum_\alpha k^\alpha E_\alpha$ and $\mathfrak{b} = \mu_*\mathcal{O}_Y(-\sum_{\rho\neq\rho_0}(r^\rho - k^\rho)E_\rho)$, it follows that

$$\mathfrak{b} \cap \mu_*\mathcal{O}_Y((k^{\rho_0} - r)E_{\rho_0}) \subset \mathfrak{b} \cap \mu_*\mathcal{O}_Y((k^{\rho_0} - r + 1)E_{\rho_0}),$$

i.e. that $\mu_*\mathcal{O}_Y((k^{\rho_0} - r)E_{\rho_0}) \subset \mu_*\mathcal{O}_Y((k^{\rho_0} - r + 1)E_{\rho_0})$. Now, set $\mathfrak{q} = \mu_*\mathcal{O}_Y(-B_{\rho_0})$. The Enriques tree associated to $\mu' : Y' \to X$, the minimal log resolution of $\mathfrak{q}$, is the path from the root to the vertex P_{ρ_0} of the Enriques tree associated to $\mathfrak{a}$. Let $\mathfrak{V}'$ be the set of vertices of this path and $\mathfrak{R}' \subset \mathfrak{R} \cap \mathfrak{V}'$ the set of relevant positions. If $\mathfrak{q} \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-\sum_{\alpha\in\mathfrak{V}'} e^\alpha E_\alpha)$, let $\mathfrak{R}'' \subset \mathfrak{R}'$ be the subset of relevant positions such that for any $\rho \in \mathfrak{R}''$, re^ρ/e^{ρ_0} is an integer. Then, using again Proposition 2.2 and the previous strict inclusion,

$$\begin{aligned}\mathcal{J}(\mathfrak{q}^{r/e^{\rho_0}}) &= \mu_*\mathcal{O}_{Y'}\left(K_{\mu'} - \sum_{\rho\in\mathfrak{R}'} \frac{re^\rho}{e^{\rho_0}} E_\rho\right) = \mu_*\mathcal{O}_{Y'}\left(\sum_{\rho\in\mathfrak{R}'}\left(k^\rho - \frac{re^\rho}{e^{\rho_0}}\right)E_\rho\right) \\ &\subset \mu_*\mathcal{O}_{Y'}\left(\sum_{\rho\in\mathfrak{R}'\setminus\mathfrak{R}''}\left(k^\rho - \frac{re^\rho}{e^{\rho_0}}\right)E_\rho + \sum_{\rho\in\mathfrak{R}''}\left(k^\rho - \frac{re^\rho}{e^{\rho_0}} + 1\right)E_\rho\right) \\ &= \mathcal{J}(\mathfrak{q}^{(1-\varepsilon)r/e^{\rho_0}}),\end{aligned}$$

where $0 < \varepsilon < 1$. Hence r/e^{ρ_0} is a jumping number of $\mu_*\mathcal{O}_Y(-B_{\rho_0})$ associated to the relevant position ρ_0. □

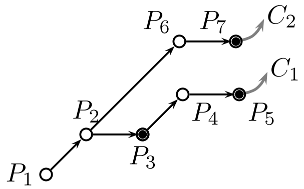

Fig. 2. Relevant positions, relevant values, and jumping numbers.

Example 2.8. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be simple complete ideals supported at P. We want to show that for a relevant position ρ, the set of relevant values of the ideal $\mu_*\mathcal{O}_Y(-B_\rho)$ are indeed needed, *i.e.* the union of the sets of relevant values of each ideal $\mathfrak{a}_i$ associated to ρ is not sufficient. Let C_1 and C_2 be two unibranch curves, general elements in $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and let the associated augmented Enriques tree of the minimal log resolution of $C_1 + C_2$ be as in the Fig. 2. We have $\mu^*(C_1 + C_2) = \widetilde{C}_1 + \widetilde{C}_2 + 6E_1 + 10E_2 + 18E_3 + 19E_4 + 38E_5 + 11E_6 + 22E_7$. The relevant positions are indicated by the black vertices: 3, 5 and 7. The jumping numbers of $\mathfrak{a}_1$ contributed by E_3 are $(5 + 6k)/12$, with $k \in \mathbb{N}$. But the three first jumping numbers of $\mathfrak{a}_1\mathfrak{a}_2$ are $5/18$, $7/18$ and $8/18$. Hence the relevant values 7 and 8 are not among the relevant values of $\mathfrak{a}_1$ associated to the relevant position 3. Of course, the well known jumping numbers of the ideal $\mu_*\mathcal{O}_Y(-B_3)$ are $(2a + 3b)/6$, with a and b positive integers.

3. Abelian coverings of smooth projective surfaces

In this section, we summarize the definition and some properties of the standard abelian coverings of X, a smooth projective surface, in a form convenient for the proof of Theorem 4.2 in the next section. We also introduce the partition of the branch curve induced by a standard abelian covering, and the distinguished edges of a curve endowed with a D-partition.

Abelian coverings

Let X be a smooth projective surface. Following [22], we recall the characterization of the normal abelian Galois ramified coverings of X.

Let S be a normal projective surface and let $\pi : S \to X = S/G$ be a Galois covering with abelian Galois group G. It is well known that $\pi_*\mathcal{O}_S$ is a coherent sheaf of $\mathcal{O}_X$-algebras and that $S \simeq \mathbf{Spec}_{\mathcal{O}_X}(\pi_*\mathcal{O}_S)$. By the theorem on the purity of the branch locus, the critical set of π is a divisor R—the ramification divisor. Its image, $C = \pi(R)$, is called the branch divisor, or branch curve. Moreover, π if flat and hence $\pi_*\mathcal{O}_S$ is locally free of rank n the order of G. The action of G on $\pi_*\mathcal{O}_S$ decomposes it into the direct sum of eigenline bundles associated to the characters $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$:

$$\pi_*\mathcal{O}_S = \bigoplus_{\chi \in \widehat{G}} \mathcal{L}_\chi{}^{-1}.$$

The action of G on $\mathcal{L}_\chi{}^{-1}$ is the multiplication by the character χ. Note that $\mathcal{L}_1 \cong \mathcal{O}_X$.

To describe the ring structure of $\pi_*\mathcal{O}_S$, *i.e.* the $\mathcal{O}_X$-linear maps ${\mathcal{L}_\chi}^{-1} \otimes {\mathcal{L}_{\chi'}}^{-1} \to {\mathcal{L}_{\chi\chi'}}^{-1}$, and hence to know how to construct abelian coverings of X, we need some notation. Let D be an irreducible component of the ramification divisor R. The curve D is associated to its *inertia subgroup* $H \subset G$ and to a character $\psi \in \widehat{H}$ that generates $\widehat{H}$ (see [22, Lemma 1.1]). Of course, H is the subset of elements of G that globally fix D and $\widehat{H}$ is the group of unitary characters of H. The character ψ corresponds to the induced representation of H on the cotangent space to S at D. Dualizing the inclusion $H \subset G$, such a pair (H, ψ) is equivalent to a group epimorphism $f \colon \widehat{G} \to \mathbb{Z}/m_f$, where $m_f = |H|$. Using this epimorphism, the induced representation $\chi\,|_H$, $\chi \in \widehat{G}$, is given by $\psi^{f(\chi)^\bullet}$. Recall that $a^\bullet$ denotes the smallest non-negative integer in the equivalence class of $a \in \mathbb{Z}/m$. We denote by $B_f \subset X$ the subdivisor of the branch locus defined set-theoretically as $\pi(R_f)$, where R_f is the union of all the components of the ramification locus associated to the group epimorphism f. Denoting by $\mathfrak{F}$ the set of all group epimorphisms from $\widehat{G}$ to the different cyclic groups $\mathbb{Z}/m\mathbb{Z}$, we have, set-theoretically, that the branch curve satisfies

$$C = \bigcup_{f\in\mathfrak{F}} B_f.$$

Let $\chi_1, \ldots, \chi_s \in \widehat{G}$ such that the group of characters $\widehat{G}$ is the direct sum of the cyclic subgroups generated by $\chi_1, \ldots, \chi_s$. Let $n_1, \ldots, n_s$ be their orders. In [22, Proposition 2.1] it is shown that the ring structure of $\pi_*\mathcal{O}_S$ is determined by the following isomorphisms:

$$\mathcal{L}_{\chi_j}^{n_j} \cong \bigotimes_{f\in\mathfrak{F}} \mathcal{O}_X\left(\frac{n_j f(\chi_j)^\bullet}{m_f} B_f\right), \quad 1 \le j \le s.$$

Note that each coefficient $n_j f(\chi_j)^\bullet/m_f$ is an integer. Fixing, for every character χ, a divisor L_χ associated to the eigensheaf $\mathcal{L}_\chi$, *i.e.* $\mathcal{L}_\chi \cong \mathcal{O}_X(L_\chi)$, these isomorphisms become

$$n_j L_{\chi_j} \sim \sum_{f\in\mathfrak{F}} \frac{n_j f(\chi_j)^\bullet}{m_f} B_f, \quad 1 \le j \le s. \tag{5}$$

Definition 3.1 (*See [22]*)**.** The line bundles $\mathcal{L}_{\chi_j}$, $1 \le j \le s$, and the divisors B_f, $f \in \mathfrak{F}$, are called a set of *reduced building data* for the covering $S \to X$.

In case X is compact, the isomorphisms (5) uniquely determine the covering up to isomorphisms of abelian coverings. It is to be noticed that if $\chi = \chi_1^{a_1} \cdots \chi_s^{a_s} \in \widehat{G}$, then

$$L_\chi \sim \sum_{j=1}^{s} a_j L_{\chi_j} - \sum_{f\in\mathfrak{F}} \left\lfloor \sum_{j=1}^{s} \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor B_f. \tag{6}$$

Hence, to a normal covering $\pi : S \to X$ of X, a set of reduced building data is associated satisfying the relations (5). Conversely, starting with a set of reduced building data and relations (5), an abelian covering may be constructed. Such a covering will be called a *standard abelian covering*.

Before we go on, we collect three remarks for further use and introduce the partition of the branch curve of a standard abelian covering.

Remark 3.2. (i) The set of irreducible components of the branch curve is endowed with the following equivalence relation: B is equivalent to B' if and only if for each $f \in \mathfrak{F}$, either both B and B' are components of B_f, or both are not. The partition corresponding to this equivalence relation is said to be the *partition associated to the standard abelian cover*. The partition will be denoted by $\boldsymbol{C} = (\ldots, C_i, \ldots)$, where each C_i is the sum of the irreducible components of an equivalence class. Clearly, the branch curve C satisfies

$$C = \sum_i C_i.$$

(ii) Since each coefficient $n_j f(\chi_j)^\bullet / m_f$ is an integer, the linear equivalences (6) defining a standard abelian covering lead, using the partition of the branch curve, to

$$n_j L_{\chi_j} \sim \sum_i \mu_j^i C_i, \tag{7}$$

where each μ_j^i is an integer.

(iii) The standard abelian covering $S \to X$ defined by the linear equivalences (5) may not be normal. In [22, Section 3] an explicit algorithm to compute the building data of the normalization of S in terms of the initial building data is described. Using this algorithm, let $S' \xrightarrow{\pi'} X$ be the normal abelian covering thus obtained. Then, by [20, Proposition 3.2] and Proposition A.1, if $\chi = \chi_1^{a^1} \cdots \chi_s^{a^s}$, the eigenline bundles of $\pi'_* \mathcal{O}_{S'}$ are given by

$$L'_\chi \sim \sum_j a^j L_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu_j^i}{n_j} \right\rfloor C_i. \tag{8}$$

In particular, the linear equivalences (7) may characterize more than one standard abelian covering of X. But all these coverings have a common normal model which is a standard abelian covering of X.

Example 3.3. In the literature, abelian coverings appear as a result of two main constructions. In the first one, the building data for a standard abelian covering over $\mathbb{P}^2$ are used. Let H be a smooth ample divisor. Consider $C_0, C_1, \ldots, C_t$ reduced curves on X having no common irreducible components such that

$$C_i \sim d_i H \quad \text{for each } i = 0, \ldots, t$$

with $d_0 = 1$. Set $G = \oplus_{j=1}^s \mathbb{Z}/n_j\mathbb{Z}$ and

$$\mathcal{L}_{\chi_j} = \mathcal{O}_X \left(\left\lceil \sum_{i=1}^t \mu_j^i d_i / n_j \right\rceil H \right),$$

where $(\chi_1, \ldots, \chi_s)$ is the canonical set of generators for $\widehat{G}$. If $M = [\mu_j^i]$ is an $(t+1) \times s$ matrix with non negative integer entries such that

$$\mu_j^0 = \left\lceil \frac{1}{n_j} \sum_{i=1}^t \mu_j^i d_i \right\rceil n_j - \sum_{i=1}^t \mu_j^i d_i \quad \text{for each } j,$$

then the linear equivalences

$$n_j L_{\chi_j} \sim \sum_{i=0}^{t} \mu_j^i C_i, \quad 1 \le j \le s,$$

define a standard abelian covering ramified over the curve $C_0 + \cdots + C_t$.

The second construction uses the existence theorem of Grauert and Remmert in [5] (see *e.g.* [6]) and is based on the composition of the Hurewich epimorphism with the morphism given by the change of constants in the homology groups

$$\pi_1(U) \longrightarrow H_1(U, \mathbb{Z}) \longrightarrow H_1(U, \mathbb{Z}/n\mathbb{Z}).$$

The open set $U = \mathbb{P}^2 \setminus (C \cup H_0) = \mathbb{C}^2 \setminus C$, with $C = \sum_{i=1}^{s} C_i$ the decomposition of C into *irreducible components* of degree d_i and H_0 a line transverse to C. It is known that there exists an exact sequence (see [2, Proposition 1.3])

$$\mathbb{Z} \xrightarrow{\iota} \bigoplus_{j=0}^{s} \mathbb{Z} \longrightarrow H_1(U, \mathbb{Z}) \longrightarrow 0$$

with $\iota(1) = g_0 + \sum_{i=1}^{s} d_i g_i$, $g_0 = (1, 0, \ldots)$ and so on. It follows that the epimorphism corresponds to a Galois unbranched covering $V \to U$ with group $H_1(U, \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^s$. By the existence theorem of Grauert and Remmert [5], this covering extends to a unique normal abelian covering $\pi : \Sigma_n \to \mathbb{P}^2$, *i.e.* such that $\pi^{-1}(U) = V$. It turns out that Σ_n is a particular case of the above construction. More precisely, it is the normalization of the standard $(\mathbb{Z}/n\mathbb{Z})^s$-covering of the projective plane with M the $(s+1) \times s$ matrix

$$\begin{bmatrix} \lceil d_1/n \rceil n - d_1 & \cdots & \lceil d_s/n \rceil n - d_s \\ & I_s & \end{bmatrix}.$$

Guided by the previous example, we give the following definition and introduce a notation.

Definition 3.4. Let C be a reduced curve of X. A partition $\boldsymbol{C} = (C_1, \ldots, C_t)$ of C is called a D-partition if there exist a divisor D and positive integers $d_1, \ldots, d_t$ such that $C_i \sim d_i D$ for each i.

Notation 3.5. Let X be a projective smooth surface. Let $\boldsymbol{C} = (C_1, \ldots, C_t)$ be a D-partition, $\boldsymbol{n} = (n_1, \ldots, n_s)$, and M be a $t \times s$ matrix with non negative integer entries. If for each j, $\sum_i \mu_j^i d_i$ is divisible by n_j, then $S(\boldsymbol{n}, M, \boldsymbol{C}, X)$ is the normal standard abelian covering corresponding to

$$n_j L_{\chi_j} \sim \sum_{i=1}^{t} \mu_j^i C_i, \quad 1 \le j \le s.$$

The notation suggests that the curves C_j of the partition $\boldsymbol{C}$ play a similar role in a formula for the irregularity. We will see in Theorem 4.2 and in Corollary 4.3 that this is indeed the case even if the two constructions mentioned above might indicate the opposite: a curve is chosen and the construction is performed using it—the line at infinity for the projective plane for example. We will see that in the formula for the irregularity, the curve becomes indistinguishable from the other components of the partition; moreover, if it has only nodes and intersect transversely the others, it plays no role at all.

Jumping walls and distinguished edges of a curve endowed with a D-partition

Let C be a reduced curve of the smooth surface X endowed with the partition $\boldsymbol{C} = (C_1, \ldots, C_t)$. Clearly, $C = \sum_i C_i$. For each singular point of C, we shall need to consider the mixed multiplier ideal $\mathcal{J}(\boldsymbol{x} \cdot \boldsymbol{C}) = \mathcal{J}(x^1 C_1 + \cdots + x^t C_t)$ with $\boldsymbol{x}$ varying in the hypercube $[0, 1)^t$. Interpreting the results of Section 2 in this context, it follows that the jumping walls associated to these multiplier ideals cut up the hypercube into convex rational polytopes on which the map $x \mapsto \mathcal{J}(\boldsymbol{x} \cdot \boldsymbol{C})$ is constant. Note that the fibres of this map are neither open nor closed.

Definition 3.6. An *edge*[2] associated to C endowed with the partition $\boldsymbol{C}$ is a finite intersection of jumping walls and coordinate hyperplanes.

Remark 3.7. Even if we are interested in the jumping walls intersecting the hypercube, the context of rational divisors is not sufficient, since the jumping walls are determined by relevant values associated to ideal sheaves—more precisely, a jumping wall might intersect a coordinate axis in a jumping number bigger than 1 associated only to an ideal sheaf.

Notation 3.8. If W is an edge associated to a curve C endowed with the partition $\boldsymbol{C}$, the set $\mathcal{U}(W)$ is the set of the connected components of the difference between W and the union of all the jumping walls and coordinate hyperplanes that do not contain W. The mixed multiplier ideal is constant on each $U \in \mathcal{U}(W)$ and will be denoted by $\mathcal{J}(U \cdot \boldsymbol{C})$.

If $\boldsymbol{C}$ is a D-partition and if $\boldsymbol{d}$ is the vector $(d_1, \ldots, d_t)$, where $\deg C_i \sim d_i D$, we define the height function $h_{\boldsymbol{C},D} : \mathbb{R}^t \to \mathbb{R}$ by $h_{\boldsymbol{C},D}(\boldsymbol{x}) = \boldsymbol{d} \cdot \boldsymbol{x}$. If no confusion is likely, we will denote the height function by $h_{\boldsymbol{C}}$.

Definition 3.9. An edge W of the projective curve C endowed with the D-partition $\boldsymbol{C}$ is called a *distinguished edge* if the height function $h_{\boldsymbol{C}}$ is constant on W. The set of distinguished edges will be denoted by $\mathcal{F}_D(\boldsymbol{C})$ or $\mathcal{F}(\boldsymbol{C})$.

Lemma 3.10. *Let W be a distinguished edge of $C \subset X$ endowed with the D-partition $\boldsymbol{C}$. For each i, either $x^i = 0$ along W, or C_i passes through P, a singular point of C to which one of the jumping walls that cut out W is associated.*

Proof. It is easy to see that a distinguished edge W, seen as a subset in the first orthant of $\mathbb{R}^t$, is bounded for the euclidean metric. Suppose that $W \not\subset \{x^i = 0\}$. Then the component C_i must satisfy the conclusion since otherwise the corresponding coordinate x^i would be unbounded. $\square$

Example 3.11. Let $C_1, \ldots, C_6$ be the lines of Ceva's arrangement, a complete quadrangle $C = \sum C_i$; C_i and C_j intersect in a node of the arrangement if and only if $i + j = 7$.

[2] Libgober kindly pointed out to me that the word *wall*, used in a previous version of the paper, was misleading in codimension ≥ 2.

Ceva's arrangement has four triple points: $C_4 \cap C_5 \cap C_6$, $C_1 \cap C_2 \cap C_4$, $C_2 \cap C_3 \cap C_6$ and $C_1 \cap C_3 \cap C_5$. For C endowed with the partition $\boldsymbol{C} = (C_1, \ldots, C_6)$ there are five distinguished edges: one for each triple point and one for all four. Clearly for each triple point P there is a distinguished edge W_P; for example if $P = C_4 \cap C_5 \cap C_6$ then W_P is defined by $x^4 + x^5 + x^6 = 2$, $x^1 = x^2 = x^3 = 0$ and $h_{\boldsymbol{C}}(W_P) = 2$. Now, if W is a distinguished edge different from the W_P, then let $\varphi_\alpha(\boldsymbol{x}) = 2$ be the equations defining the jumping walls that cut out W—2 is the only relevant value. Note that each equation is of the form $x^i + x^j + x^k = 2$. Let $I \subset \{1, 2, \ldots, 6\}$ be the set of subscripts appearing in the equations φ_α. Since W is distinguished, $x^j = 0$ along W for every $j \notin I$. Furthermore the equation

$$\sum_{i \in I} x^i = h_{\boldsymbol{C}}(W)$$

is a linear combination of the φ_α. Hence, there exist reals ζ_α such that

$$\sum_\alpha \zeta_\alpha (\varphi_\alpha(\boldsymbol{x}) - 2) = \sum_{i \in I} x^i - h_{\boldsymbol{C}}(W)$$

for any $\boldsymbol{x} \in \mathbb{R}^6$. Hence $2 \sum_\alpha \zeta_\alpha = h_{\boldsymbol{C}}(W)$, and taking $x^i = 1$ for every $i \in I$, $3 \sum_\alpha \zeta_\alpha = |I|$. It follows that $2|I| = 3h_{\boldsymbol{C}}(W)$, *i.e.* $|I| = 6$ and $h_{\boldsymbol{C}}(W) = 4$. To see that W is unique with these properties it is sufficient to notice that W is defined by the four equations corresponding to the four triple points. It is clear that it should be defined by at least three out of four equations. Summing these three equations and using $\sum_1^6 x^i = 4$ we get the fourth.

Example 3.12. Let Γ_1 and Γ_2 be two conics that have common tangents at the two points of intersection P and Q. Let H_∞ be the line through P and Q. We want to determine the set of distinguished edges $\mathcal{F}$ for the curve $C = \Gamma_1 + \Gamma_2 + H_\infty$, with the H-partition $\boldsymbol{C} = (B, H_\infty)$, where $B = \Gamma_1 + \Gamma_2 \sim 4H$. The curve B has two tacnodes at P and Q, a unique jumping number $3/4 < 1$ and hence a unique relevant value 3. The branch curve C has two singular points and the exceptional configuration of the minimal log-resolution is $(2+1)E_1 + (4+1)E_2$. There are two jumping values, $3/5$ and $4/5$ and two relevant values 3 and 4 associated to the second exceptional divisor in the log-resolution for each singular point. It follows that there are two jumping walls W_3 and W_4 defined by $4x + x^\infty = 3$ and $4x + x^\infty = 4$ respectively. There are three edges and all three are distinguished since $h_{\boldsymbol{C}} = 4x + x^\infty$: W_3, W_4, and the intersection of W_3 with the coordinate line $\{x^0 = 0\}$, *i.e.* the point W_0 of coordinates $(3/4, 0)$. Finally,

$$\mathcal{U}(W_0) = \{W_0\}, \qquad \mathcal{U}(W_3) = \{W_3 \setminus W_0\} \quad \text{and} \quad \mathcal{U}(W_4) = \{W_4\}.$$

4. The irregularity of standard abelian coverings of smooth projective surfaces

In this section X is a smooth projective surface, G is the group $\oplus_{j=1}^s \mathbb{Z}/n_j\mathbb{Z}$, and $(\chi_1, \ldots, \chi_s)$ is the canonical system of generators of $\widehat{G}$.

Let $S \xrightarrow{\pi} X$ be a standard G-abelian covering with $\boldsymbol{C} = (C_1, \ldots, C_t)$ the associated partition of the branch curve $C = \sum_i C_i$ (see Remark 3.2(i)). If S is *normal*, we start by reproving Libgober's formula for the irregularity of a smooth model of S; we refer the reader to [17,1] for different arguments. Then, when $\boldsymbol{C}$ is an H-partition, with H ample, we prove Theorem 4.2 that gives the explicit formula (11) for the irregularity of a smooth model of S. It is the main result of the paper. It has to be noticed that for this formula, we do not need to suppose that the standard abelian covering S is normal. The explicit formula (11) can be used to compute the irregularity of various examples of abelian coverings appearing in the literature. We will develop some of these computations in the next section.

Theorem 4.1. *Let S be a normal standard G-covering of X with associated partition $\boldsymbol{C}$, defined by*

$$n_j L_{\chi_j} \sim \sum_{i=1}^{t} \mu_j^i C_i \quad \text{for each } j.$$

If $\widetilde{S}$ is a desingularization of S, then

$$q(\widetilde{S}) = \sum_{\chi=\chi_1^{a^1}\cdots\chi_s^{a^s}\in\widehat{G}} h^1\left(X, \omega_X \otimes \mathcal{L}_\chi \otimes \mathcal{J}\left(\sum_i \left\langle \sum_j \frac{a^j \mu_j^i}{n_j} \right\rangle C_i\right)\right).$$

Proof. Since S is normal, we have noticed in Remark 3.2(iii) that S is completely characterized by the linear equivalences $n_j L_{\chi_j} \sim \sum_{i=1}^t \mu_j^i C_i$. Moreover, by (8), since S is normal,

$$L_\chi \sim \sum_j a^j L_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu_j^i}{n_j} \right\rfloor C_i, \tag{9}$$

with $\chi = \chi_1^{a^1} \cdots \chi_1^{a^s}$.

Let $\mu : Y \to X$ be a log resolution of the branch divisor. The abelian covering $S \to X$ pulls back to a standard abelian covering $S' \to Y$ defined by line bundles $\mathcal{L}'_{\chi_j} = \mu^*\mathcal{L}_{\chi_j}$, with

$$L'_{\chi_j} \sim \sum_{i=1}^{t} \mu_j^i \widetilde{C}_i + \sum_{i,P} \mu_i^j \boldsymbol{e}_i^P \cdot \boldsymbol{E}_P,$$

where

$$\mu^* C_i = \widetilde{C}_i + \sum_P \boldsymbol{e}_i^P \cdot \boldsymbol{E}_P. \tag{10}$$

The sum in (10) is taken over all the singular points of $C = \sum_i C_i$ but the nodes; $\boldsymbol{e}_i^P \cdot \boldsymbol{E}_P$ denotes the sum

$$\sum_\alpha e_i^{P,\alpha} E_{P,\alpha},$$

where the curves $E_{P,\alpha}$ are the irreducible components of the exceptional configuration of the log resolution $\mu : Y \to X$ over P. Using the normalization procedure in [22], we obtain the following diagram.

$$\begin{tikzcd} S'' \arrow[r] \arrow[d, "\pi"] & S' \arrow[r] \arrow[d] & S \arrow[d] \\ Y \arrow[r, equal] & Y \arrow[r, "\mu"] & X \end{tikzcd}$$

The normalization procedure yields the normal surface S'' with only Hirzebruch–Jung singularities. By [20, Proposition 3.2] and Proposition A.1, if $\chi = \chi_1^{a^1} \cdots \chi_s^{a^s}$, then

$$\begin{aligned} L''_\chi &\sim \sum_j a^j L'_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu^i_j}{n_j} \right\rfloor \widetilde{C}_i - \sum_P \left\lfloor \sum_{i,j} \frac{a^j \mu^i_j}{n_j} \boldsymbol{e}^P_i \right\rfloor \cdot \boldsymbol{E}_P \\ &\sim \sum_j a^j \mu^* L_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu^i_j}{n_j} \right\rfloor \mu^* C_i \\ &\quad - \left(\sum_P \left\lfloor \sum_{i,j} \frac{a^j \mu^i_j}{n_j} \boldsymbol{e}^P_i \right\rfloor - \sum_{P,i} \left\lfloor \sum_j \frac{a^j \mu^i_j}{n_j} \right\rfloor \boldsymbol{e}^P_i \right) \cdot \boldsymbol{E}_P \\ &= \sum_j a^j \mu^* L_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu^i_j}{n_j} \right\rfloor \mu^* C_i - \sum_P \left\lfloor \sum_i \left\langle \sum_j \frac{a^j \mu^i_j}{n_j} \right\rangle \boldsymbol{e}^P_i \right\rfloor \cdot \boldsymbol{E}_P \\ &\sim \mu^* L_\chi - \sum_P \left\lfloor \sum_i \left\langle \sum_j \frac{a^j \mu^i_j}{n_j} \right\rangle \boldsymbol{e}^P_i \right\rfloor \cdot \boldsymbol{E}_P . \end{aligned}$$

For the last linear equivalence we use (9). We recall that $\langle x \rangle$ denotes the fractional part of the real number x, *i.e.* $x = \lfloor x \rfloor + \langle x \rangle$.

Let $\widetilde{S}$ be a desingularization of S''. Using the Leray spectral sequence and the Serre duality, we have

$$q(\widetilde{S}) = h^1(S'', \mathcal{O}_{S''}) = h^1(Y, \pi_* \mathcal{O}_{S''}) = \sum_{\chi \in \widehat{G}} h^1(Y, \omega_Y \otimes \mathcal{L}''_\chi).$$

Hence, using the above expression for L''_χ,

$$\begin{aligned} q(\widetilde{S}) &= \sum_{\chi \in \widehat{G}} h^1 \left(Y, \omega_Y \otimes \mu^* \mathcal{L}_\chi \otimes \mathcal{O}_Y \left(- \sum_P \left\lfloor \sum_i \left\langle \sum_j \frac{a^j \mu^i_j}{n_j} \right\rangle \boldsymbol{e}^P_i \right\rfloor \cdot \boldsymbol{E}_P \right) \right) \\ &= \sum_{\chi = \chi_1^{a^1} \cdots \chi_s^{a^s} \in \widehat{G}} h^1 \left(X, \omega_X \otimes \mathcal{L}_\chi \otimes \mathcal{J} \left(\sum_i \left\langle \sum_j \frac{a^j \mu^i_j}{n_j} \right\rangle C_i \right) \right). \quad \square \end{aligned}$$

Now, let us suppose that the partition $\boldsymbol{C} = (C_1, \ldots, C_t)$ associated to $S \to X$ is an H-partition, with H ample, and that S is the normal surface defined by the linear equivalences

$$n_j L_{\chi_j} \sim \sum_{i=1}^t \mu^i_j C_i, \quad 1 \le j \le s,$$

where, for each j, the integer $\sum_i \mu^i_j d_i$ is a multiple of n_j and

$$\mathcal{L}_{\chi_j} = \mathcal{O}_X\left(\frac{1}{n_j}\sum_{i=1}^{t}\mu^i_j d_i\, H\right).$$

Using the notation introduced in Example 3.3, $S = S(\boldsymbol{n}, M, \boldsymbol{C}, X)$. We can state the main result of the paper.

Theorem 4.2. *Let $S = S(\boldsymbol{n}, M, \boldsymbol{C}, X)$ with $\boldsymbol{C} = (C_1, \ldots, C_t)$ an H-partition, with H-ample. If $\widetilde{S}$ is a desingularization of S, then*

$$q(\widetilde{S}) = q(X) + \sum_{W\in\mathcal{F}(\boldsymbol{C})}\sum_{U\in\mathcal{U}(W)} |U|^M_{\boldsymbol{n}}\cdot h^1 \times \big(X, \omega_X\otimes\mathcal{O}_X(h_{\boldsymbol{C}}(W)H)\otimes\mathcal{J}(U\cdot\boldsymbol{C})\big), \tag{11}$$

where, for any rational convex polytope $U\subset\mathbb{R}^t$,

$$|U|^M_{\boldsymbol{n}} = \operatorname{card}\varphi^{-1}(U\cap[0,1)^t),$$

and the map $\varphi : [0,1)^s\cap\bigoplus_{j=1}^s 1/n_j\mathbb{Z}\longrightarrow[0,1)^t$ (depending on $\boldsymbol{n}\in\mathbb{N}^s$ and the $t\times s$-matrix M) is defined by

$$\varphi\left(\frac{a^1}{n_1},\ldots,\frac{a^s}{n_s}\right) = \left(\left\langle\sum_j \mu^1_j\frac{a^j}{n_j}\right\rangle,\ldots,\left\langle\sum_j\mu^t_j\frac{a^j}{n_j}\right\rangle\right).$$

Proof. Using the definition of S,

$$n_j\deg_H(\mathcal{L}_{\chi_j}) = \sum_{i=0}^{t}\mu^i_j d_i, \tag{12}$$

where $\deg_H D$ is the integer defined by the linear equivalence $D\sim(\deg_H D)H$. If $\chi = \chi_1^{a^1}\cdots\chi_s^{a^s}$, $0\le a^j<n_j$, then, by (9) and (12),

$$\begin{aligned} L_\chi &\sim \sum_{j=1}^s a^j L_{\chi_j} - \sum_{i=1}^t\left\lfloor\sum_{j=1}^s\frac{a^j\mu^i_j}{n_j}\right\rfloor C_i\\ &\sim\left(\sum_{j=1}^s a^j\frac{1}{n_j}\sum_{i=1}^t\mu^i_j d_i - \sum_{i=1}^t\left\lfloor\sum_{j=1}^s\frac{a^j\mu^i_j}{n_j}\right\rfloor d_i\right)H\\ &= \left(\sum_{i=1}^t\left\langle\sum_{j=1}^s\frac{a^j\mu^i_j}{n_j}\right\rangle d_i\right)H.\end{aligned}$$

Setting, for every $1\le i\le t$,

$$x^i = \left\langle\sum_{j=1}^s\frac{a^j\mu^i_j}{n_j}\right\rangle, \tag{13}$$

it follows that

$$L_\chi \sim \sum_{i=1}^{t} x^i d_i\, H = h_{\boldsymbol{C}}(\boldsymbol{x})H, \qquad h_{\boldsymbol{C}}(\boldsymbol{x}) \in \mathbb{Z}. \tag{14}$$

Finally, by Theorem 4.1,

$$\begin{aligned} &h^1\left(X, \omega_X \otimes \mathcal{L}_\chi \otimes \mathcal{J}\left(\sum_{i=1}^{t}\left\langle \sum_{j=1}^{s} \frac{a^j \mu_j^i}{n_j}\right\rangle C_i\right)\right) \\ &\quad = h^1\big(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}}(\boldsymbol{x})H) \otimes \mathcal{J}(\boldsymbol{x}\cdot\boldsymbol{C})\big). \end{aligned} \tag{15}$$

By (14), $h_{\boldsymbol{C}}(\boldsymbol{x})$ is an integer; hence the h^1 in (15) might be non-zero whenever $\boldsymbol{x} = (x^1, \ldots, x^t)$ satisfies two conditions. First, for every curve C_i such that $x^i \neq 0$, there exist a singular point P of C lying on C_i and a relevant position α of P such that the number $\sum_{i=1}^t x^i e_i^{P,\alpha}$ is a relevant value of $\boldsymbol{C}$ at (P, α). Indeed, if this condition does not hold for C_1 for example, it is sufficient to notice that $\mathcal{J}(D') = \mathcal{J}(\boldsymbol{x}\cdot\boldsymbol{C})$, where $D' = (x^1 - \varepsilon)C_1 + \sum_{i\neq 1} x^i C_i$ for $\varepsilon > 0$ sufficiently small, and to apply the Kawamata–Viehweg–Nadel vanishing theorem to see that the right-hand side of (15) vanishes. Second, suppose that the first condition is fulfilled, *i.e.* for every component C_i with $x^i \neq 0$ there exists a singular point P of the branch curve C and a position α such that $r_{P,\alpha} = \sum_i x^i e_i^{P,\alpha}$ is a relevant value of $\boldsymbol{x}\cdot\boldsymbol{C}$ at (P, α). If W is the space of solutions of these equations seen as equations in the unknowns x^i, then W is an edge for the partition $\boldsymbol{C} = (C_1, \ldots, C_t)$ and the linear operator $h_{\boldsymbol{C}} : \boldsymbol{x} \mapsto \sum_{i=1}^t x^i d_i$ must be a constant on W. Indeed, if W is positive dimensional and not contained into a fibre of $h_{\boldsymbol{C}}$, it is sufficient to take $\boldsymbol{y} \in W$ such that $\sum_i y^i d_i < \sum_i x^i d_i$ and $\lceil \sum_i y^i d_i \rceil = \sum_i x^i d_i$; then, using again the Kawamata–Viehweg–Nadel vanishing theorem,

$$\begin{aligned} &h^1(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}}(\boldsymbol{x})H) \otimes \mathcal{J}(\boldsymbol{x}\cdot\boldsymbol{C})) \\ &\quad = h^1(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}}(\boldsymbol{x})H) \otimes \mathcal{J}(\boldsymbol{y}\cdot\boldsymbol{C})) = 0. \end{aligned}$$

So the edge W is distinguished and $\mathcal{J}(\boldsymbol{x}\cdot\boldsymbol{C}) = \mathcal{J}(U\cdot\boldsymbol{C})$, for U the corresponding connected component defined on W by the other jumping walls and coordinate hyperplanes.

By (4) and the previous considerations, we conclude that

$$q(S'') = q(X) + \sum_{W\in\mathcal{F}} \sum_{U\in\mathcal{U}(W)} |U|_{\boldsymbol{n}}^M \cdot h^1(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}}(W)H) \otimes \mathcal{J}(U\cdot\boldsymbol{C}))$$

where

$$\begin{aligned} |U|_{\boldsymbol{n}}^M = \operatorname{card}\Bigg\{ &(a^1, \ldots, a^s) \mid 0 \le a^j < n_j \text{ for every } j, \text{ and} \\ &\left(\left\langle \sum_j a^j \mu_j^1/n_j \right\rangle, \ldots, \left\langle \sum_j a^j \mu_j^t/n_j \right\rangle\right) \in U \Bigg\}. \quad \square \end{aligned} \tag{16}$$

Remark. If the surface X is not normal, a similar but direct and slightly longer calculation, without invoking Theorem 4.1, leads to the same result.

Using Lemma 3.10, or detailing more the above argument, we deduce from Theorem 4.2 a formula for the irregularity which makes use of the significant curves of the partition $\boldsymbol{C}$. A curve C_i of the partition $\boldsymbol{C}$ is called *significant* if it contains a singularity of C that is not a node.

Corollary 4.3. *Let $S = S(\boldsymbol{n}, M, \boldsymbol{C}, X)$ with $\boldsymbol{C}$ an H-partition, H-ample, and let $\widetilde{S}$ be a desingularization of S. If*

$$\boldsymbol{C}^* = (C_i \mid C_i \text{ significant})$$

and M^* is the sub-matrix of M formed by the lines corresponding to $\boldsymbol{C}^*$, then

$$q(\widetilde{S}) = q(X) + \sum_{W \in \mathcal{F}(\boldsymbol{C}^*)} \sum_{U \in \mathcal{U}(W)} |U|_{\boldsymbol{n}}^{M^*} \cdot h^1 \times \big(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}^*}(W)H) \otimes \mathcal{J}(U \cdot \boldsymbol{C}^*)\big).$$

5. Applications and examples

Asymptotic behaviour of the irregularity

In setting out to look for applications of Theorem 4.2 it seems best to start with the asymptotic behaviour of the irregularity of the abelian coverings of the projective plane described by Hironaka in [6]. See also [1, Theorem 1.7], where Budur establish the quasi-polynomial behaviour of the Hodge numbers $h^{0,q}$ of the finite abelian coverings of a smooth n-dimensional variety.

Theorem 5.1. *Let $S_n = S(\boldsymbol{n}, M, \boldsymbol{C}, X)$ be a $(\mathbb{Z}/n\mathbb{Z})^s$-abelian covering where $\boldsymbol{n} = (n, \ldots, n)$, $\boldsymbol{C} = (C_1, \ldots, C_t)$ is an H-partition with H-ample and $C_1 \sim H$. If*

$$\mu_j^1 = \left\lceil \frac{1}{n} \sum_{i=2}^{t} \mu_j^i d_i \right\rceil n - \sum_{i=2}^{t} \mu_j^i d_i \quad \text{and} \quad L_{\chi_j} \sim \frac{1}{n} \sum_{i=1}^{t} \mu_j^i d_i H,$$

then $q(\widetilde{S}_n)$ is a quasi-polynomial function of n of degree $\leq s$, where $\widetilde{S}_n$ is a desingularization of S_n.

Definition. A function $f : \mathbb{N} \to \mathbb{N}$ is called a quasi-polynomial function if there exist an integer $N > 0$ and polynomials $P_0, \ldots, P_{N-1}$ such that $f(n) = P_j(N)$ if $n = j \mod N$.

Proof. Let t be the length of $\boldsymbol{C}$. By Theorem 4.2,

$$q(\widetilde{S}_n) = q(X) + \sum_{W \in \mathcal{F}(\boldsymbol{C})} \sum_{U \in \mathcal{U}(W)} |U|_{\boldsymbol{n}}^{M} \cdot h^1 \times \big(X, \omega_X \otimes \mathcal{O}_X(h_{\boldsymbol{C}}(W)H) \otimes \mathcal{J}(U \cdot \boldsymbol{C})\big),$$

where, for any convex polytope $\mathcal{P} \subset \mathbb{R}^t$,

$$|\mathcal{P}|_{\boldsymbol{n}}^M = \operatorname{card} \varphi^{-1}(\mathcal{P} \cap [0,1)^t),$$

and the map $\varphi : [0,1)^s \cap \bigoplus_{j=1}^s 1/n\mathbb{Z} \longrightarrow [0,1)^t$ is defined by

$$\varphi\left(\frac{a^1}{n}, \ldots, \frac{a^s}{n}\right) = \left(\left\langle \sum_j \mu_j^1 \frac{a^j}{n} \right\rangle, \ldots, \left\langle \sum_j \mu_j^t \frac{a^j}{n} \right\rangle\right).$$

The closure of the subset $U \subset W$ in the distinguished edge W represents a convex polytope and its border in W, a finite union of convex polytopes. To end the argument, it is sufficient to show that for a convex polytope $\mathcal{P}$ in $[0,1]^t$, $|\mathcal{P}|_{\boldsymbol{n}}^M$ is a quasi-polynomial of degree $\leq s$. Consider

$$\Phi : \mathbb{R}^s \longrightarrow \mathbb{R}^t$$

the linear application given by M. Restricting to $[0,1)^s \cap \bigoplus_{j=1}^s 1/n\mathbb{Z}$ and taking the fractional part on each component of the image, we get φ. So, for the convex polytope $\mathcal{P} \subset [0,1]^t$, we consider the set of convex polytopes in $[0,1]^s$,

$$P_{\boldsymbol{k}} = \Phi^{-1}(\boldsymbol{k} + \mathcal{P}) \cap [0,1]^s, \quad \boldsymbol{k} \in \mathbb{Z}^t.$$

Clearly, $P_{\boldsymbol{k}}$ is non empty only for a finite number of $\boldsymbol{k} \in \mathbb{Z}^t$. If

$$Q_{\boldsymbol{k}} = \Phi^{-1}(\boldsymbol{k} + \mathcal{P}) \cap \left([0,1]^s \setminus [0,1)^s\right),$$

then

$$|\mathcal{P}|_{\boldsymbol{n}}^M = \sum_{\boldsymbol{k}} \left(\operatorname{card} \left(P_{\boldsymbol{k}} \cap \left(\frac{1}{n}\mathbb{Z}\right)^s \right) - \operatorname{card} \left(Q_{\boldsymbol{k}} \cap \left(\frac{1}{n}\mathbb{Z}\right)^s \right) \right).$$

Now, if $P \subset \mathbb{R}^s$ is a convex polytope, the *Ehrhart quasi-polynomial* of P is the function defined by

$$i(P,n) = \operatorname{card}(nP \cap \mathbb{Z}^s),$$

where $nP = \{n\boldsymbol{v} \mid \boldsymbol{v} \in P\}$. Clearly, the number $i(P,n)$ is equal to the number of rational points in $P \cap (1/n\mathbb{Z})^s$. We refer the reader to [24, Theorem 4.6.25] where it is shown that $i(P,n)$ is indeed a quasi-polynomial whose degree equals $\dim P$. The result follows. □

Cyclic coverings

As a particular case of Theorem 4.2 we obtain Vaquié's formula from [25] for the irregularity of cyclic coverings of a projective surface (see also [20]). This study has been initiated by Zariski in [26] where he computed the irregularity of multiple planes, in case the branch curve has only nodes and cusps as singularities and is transverse to the line at infinity. Various generalizations have since been proposed to Zariski's formula in [3,13,14, 18–20,25].

Corollary 5.2 (*See [25]*). *Let X be a smooth projective surface and let H be a smooth ample curve. Let C be a curve transverse to H and such that $C \sim dH$. If $\widetilde{S}$ is* *a desingularization of the standard $\mathbb{Z}/n\mathbb{Z}$-covering of the plane defined by the linear equivalence $nL_\chi \sim C + (\lceil d/n \rceil n - d)H$, then*

$$q(\widetilde{S}) = q(X) + \sum_{\substack{\xi \text{ jumping number of } C \\ \xi \in 1/(n\wedge d)\mathbb{Z},\ 0<\xi<1}} h^1(X, \mathcal{O}_X(K_X + \xi dH) \otimes \mathcal{J}(\xi \cdot C)).$$

Proof. Since H is smooth and transverse to C, we apply Corollary 4.3 to see that the distinguished edges in the formula (11) are points ξ in the open interval $(0, 1)$ corresponding to the jumping numbers of C such that $\xi \cdot \deg C \in \mathbb{Z}$. Moreover $|\xi|_n = \operatorname{card}(\{\xi\} \cap 1/n\mathbb{Z})$ equals 1 or 0 depending on weather $\xi n \in \mathbb{Z}$. The result follows. $\square$

If H is not transverse to C, *i.e.* both H and C are significant, then the edges in the formula (11) live in $\mathbb{R}^2$ with euclidean coordinates x and x^∞, let us say. The edges are of two types: (1) those for which $x^\infty = 0$, in which case they are jumping numbers for C and the corresponding term in (11) is determined as in the above corollary; (2) those for which $x^\infty \neq 0$ and the edge is determined by an equation whose homogeneous part must coincide with the linear form $h_{(C,H)}$ modulo the multiplication by a non-zero rational. It may be said that the results obtained for the irregularity are qualitatively different. In the transverse situation the irregularity is quasi-constant as a function of n. In the non transverse situation the irregularity depends on n. More precisely, using Hironaka's result, it is a quasi-polynomial of degree ≤ 1. The next examples illustrate this behaviour of the irregularity in the non transverse situation.

Example 5.3. Let Γ be a conic and E a cubic that intersect in six points. Let C be the plane sextic defined by $\gamma^3 + e^2 = 0$, where γ and e are equations for Γ and E respectively. The sextic C has six cusps on the conic Γ. We consider the cyclic covering $S_n \to \mathbb{P}^2$ defined by the reduced building data $\mathcal{L}_\chi = \mathcal{O}_{\mathbb{P}^2}(2)$ and C, Γ, such that

$$nL_\chi \sim C + (n-3)\Gamma.$$

In particular, for $n \neq 3$, S_n is ramified along $C + \Gamma$. The associated partition is (C, Γ), a $2H$-partition.

There are two distinguished jumping walls

$$W_k = \{(x, x^\infty) \mid 6x + 2x^\infty = k\} \quad k = 5, 7$$

since the relevant position is given by the last exceptional divisor in the log-resolution of a cusp—the log-resolution of a cusp is also the log-resolution of a singularity of $C + \Gamma$. The first two jumping numbers of a singularity of $C+\Gamma$ are $5/8$ and $7/8$. The only intersections of these jumping walls are with the coordinate hyperplane $x^\infty = 0$. Hence, the formula for the irregularity becomes

$$q(\widetilde{S}_n) = |W_5|\, h^1(\mathbb{P}^2, \mathcal{I}_Z(-3+5)) + |W_7|\, h^1(\mathbb{P}^2, \mathcal{I}_{Z'}(-3+7)),$$

where Z is the support of the six cusps and Z' is the subscheme given by the support plus the tangent directions. Now, it is well known that $h^1(\mathbb{P}^2, \mathcal{I}_Z(2)) = 1$; using the trace-residual exact sequence for E and Z' (see [7]), we obtain $h^1(\mathbb{P}^2, \mathcal{I}_{Z'}(4)) = 1$. Hence

$$q(\widetilde{S}_n) = \sum_{k=5,7} \left| \{(a, b) \mid 6a + 2b = kn, 0 \leq a, b \leq n-1\} \cap \mathbb{N}^2 \right|$$

It follows that $q(\widetilde{S}_n) = 0$ if n is odd, and if n is even,

$$\begin{aligned}
q(\widetilde{S}_n) &= \left|\left\{a \in \mathbb{N} \mid \frac{n}{2} + \frac{1}{3} \le a \le \frac{10n}{6}\right\}\right| \\
&\quad + \left|\left\{a \in \mathbb{N} \mid \frac{10n}{6} + 1 \le a \le \frac{14n}{6},\ a \le n-1\right\}\right| \\
&= \left|\left\{a \in \mathbb{N} \mid \frac{n}{2} + \frac{1}{3} \le a \le n-1\right\}\right| \\
&= \frac{n}{2} - 1.
\end{aligned}$$

Example 5.4. The two conics Γ_i, $i = 1, 2$, with common tangents at P and Q considered in Example 3.12 provide cyclic coverings with maximal degrees for the quasi-polynomials that represent the irregularity, whatever the relative position of the line at infinity, H_∞.

If H_∞ is transverse to $\Gamma_1 + \Gamma_2$, then the $\mathbb{Z}/n\mathbb{Z}$-cyclic covering

$$S_n = S\left(n, \begin{bmatrix} 1 \\ \lceil d/n \rceil n - d \end{bmatrix}, (\Gamma_1 + \Gamma_2, H_\infty), \mathbb{P}^2\right)$$

has non vanishing irregularity if and only if n is divisible by 4—3/4 is the only jumping number of B—in which case

$$q(S_n) = h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) = 1.$$

Now, if H_∞ is the line through P and Q, then we have seen in Example 3.12 that there are two jumping walls W_3 and W_4, and three distinguished edges, the previous two and the point W_0, the intersection of W_3 with the coordinate plane $x^\infty = 0$. Set $C = \Gamma_1 + \Gamma_2 + H_\infty$. By Corollary 4.3, if $n \ge 2$, we get

$$\begin{aligned}
q(S_n) &= |W_0|_n\, h^1(\mathbb{P}^2, \mathcal{J}(W_0 \cdot \boldsymbol{C})) + |W_3 \smallsetminus W_0|_n\, h^1(\mathbb{P}^2, \mathcal{J}(W_3 \smallsetminus W_0 \cdot \boldsymbol{C})) \\
&\quad + |W_4|_n\, h^1(\mathbb{P}^2, \mathcal{O}(1) \otimes \mathcal{J}(W_4 \cdot \boldsymbol{C})) \\
&= |W_0|_n\, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_3 \smallsetminus W_0|_n\, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_4|_n\, h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) \\
&= |W_3|_n\, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_4|_n\, h^1(\mathbb{P}^2, \mathcal{I}_Z(1)),
\end{aligned} \tag{17}$$

where Z is the subscheme supported at P and Q and determined by the points and the directions of the tangents to the two conics at P and Q. Since

$$|W_l|_n = \operatorname{card}\{(x, x^\infty) \mid 4x + x^\infty = l,\ 0 \le x < 1,\ 0 \le x^\infty < 1, x, x^\infty \in 1/n\mathbb{Z}\},$$

$l = 3, 4$, it follows that

$$q(\widetilde{S}_n) = \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n-1}{4} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Example 5.5. If in the previous example we consider the abelian covering Σ_n of $\mathbb{P}^2$ defined by the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and branched along $C = \Gamma_1 + \Gamma_2 + H_\infty$ with the H-partition

$\boldsymbol{C} = (\Gamma_1, \Gamma_2, H_\infty)$, the same formula for the irregularity is obtained as in (17), but this time, the edges are defined in $\mathbb{R}^3$ by

$$W_l = \{(x^1, x^2, x^\infty) \mid 2x^1 + 2x^2 + x^\infty = l\},$$

$l = 3, 4$. Then

$$|W_3|_n = \operatorname{card}\big(\{(i, j) \in \mathbb{N}^2 \mid n + 1 \le i + j \le 3n/2\} \cap [0, 1)^2\big)$$
$$|W_4|_n = \operatorname{card}\big(\{(i, j) \in \mathbb{N}^2 \mid 3n/2 + 1 \le i + j \le 2n\} \cap [0, 1)^2\big);$$

hence

$$q(\widetilde{\Sigma_n}) = \frac{(n-1)(n-2)}{2}.$$

Line arrangements with at most triple points

Next we want to point out that the formula (11) simplifies in case X is the projective plane and the branch curve is a line arrangement $\mathcal{A}$ with at most triple points.

Notation. Let W be an edge. The subarrangement $\mathcal{A}_W$ will denote the minimal subarrangement of $\mathcal{A}$ determined by the points that contribute to W. This subarrangement is unique since all points are triple points.

Theorem 5.6. *Let $\mathcal{A} = \bigcup_{j=0}^m H_j$ be a line arrangement in the projective plane with at most triple points and let H_∞ be a line either of $\mathcal{A}$ or transverse to $\mathcal{A}$. Let $s = m - 1$ in the former case and $s = m$ in the latter. If $\widetilde{S}$ is the desingularization of the standard abelian $(\mathbb{Z}/n\mathbb{Z})^s$-covering associated to $\mathcal{A}$ and defined by*

$$nL_{\chi_j} \sim (n-1)H_0 + H_j, \quad 1 \le j \le s,$$

then

$$q(\widetilde{S}) = \sum_{W \in \mathcal{F}(\mathcal{A})} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + 2/3 \deg \mathcal{A}_W) \otimes \mathcal{I}_{Z_W}\big), \tag{18}$$

where Z_W denotes the support of the cusps defining W.

Proof. For each singular point P of the arrangement, the configuration of exceptional divisors is reduced to only one divisor E_P, with $e_{\mathcal{A}}^P = \sum_{j=0}^s e_j^P = 3$. Moreover, $e_j^P = 1$ or 0 depending on whether the line H_j passes through P or not. Since 2/3 is the only jumping number smaller than 1 for a triple point, the only relevant value is 2. It follows that for any P, the elementary wall W_P is given by

$$W_P = \{(x^1, \dots, x^s) \mid e_0^P x^0 + e_1^P x^1 + \cdots + e_s^P x^s = 2\}.$$

Now, let W be a bounded edge in the formula for the irregularity. There exists a unique minimal subarrangement $\mathcal{A}_W$ determined by the points contributing to W.

Claim. $h_{\mathcal{A}}(W) = 2/3 \deg \mathcal{A}_W$.

Indeed, let I be a subset of subscripts such that $\mathcal{A}_W = \bigcup_{i\in I} H_i$. Since $h_{\mathcal{A}}$ is constant along W, the equation $\sum_{i\in I} x^i = h_{\mathcal{A}}(W)$ is a linear combination of the equations defining W in $[0,1)^{|I|}$, where $|I| = \deg \mathcal{A}_W$. More precisely, there are constants γ_P such that

$$\sum_{i\in I} x^i - h_{\mathcal{A}}(W) = \sum_P \gamma_P \left(\sum_{i\in I} e_i^P x^i - 2 \right). \tag{19}$$

Hence $h_{\mathcal{A}}(W) = 2\sum_P \gamma_P$, and setting $x^i = 1$ for all $i \in I$ in (19),

$$\deg \mathcal{A}_W - h_{\mathcal{A}}(W) = \sum_P \gamma_P (3-2).$$

The claim follows.

To end the proof of the theorem, it is sufficient, for any $U \in \mathcal{U}(W)$, to consider the subscheme Z_W of points P that are among the triple points of A_W and for which $\sum_{i\in I} e^i_P x^i = 2$. $\square$

Example 5.7 (*The Ceva Arrangement* $A_1(6)$)**.** Let $\mathcal{A}$ be the Ceva arrangement, *i.e.* a complete quadrangle. It is a line arrangement of degree 6 with three double points and four triple points. Let $\widetilde{S}_n$ be a desingularization of the abelian $(\mathbb{Z}/n\mathbb{Z})^5$-covering of $\mathbb{P}^2$ branched along $\mathcal{A}$. Then

$$q(\widetilde{S}_n) = \frac{5(n-2)(n-1)}{2}.$$

These surfaces are introduced by Hirzebruch in [8]. If $n = 5$, the irregularity was computed by Ishida in [10]. The general case was dealt with by Libgober in [15].

The sub-arrangements that may have a non-zero contribution in the formula (18) are either the pencil sub-arrangement $\mathcal{A}_P$ of a triple point P, or the arrangement $\mathcal{A}$. Now,

$$h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + 2/3 \deg \mathcal{A}_P) \otimes \mathcal{J}(2/3 \cdot \mathcal{A}_P)\big) = h^1(\mathbb{P}^2, \mathcal{I}_P(-1)) = 1$$

and, if Z denotes the support of the triple points,

$$h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + 2/3 \deg \mathcal{A}) \otimes \mathcal{J}(2/3 \cdot \mathcal{A})\big) = h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) = 1.$$

So,

$$\begin{aligned} q(\widetilde{S}_n) &= \sum_P |W(\mathcal{A}_P)| \cdot h^1(\mathbb{P}^2, \mathcal{I}_P(-1)) + |W(\mathcal{A})| \cdot h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) \\ &= \sum_P |W(\mathcal{A}_P)| + |W(\mathcal{A})|. \end{aligned}$$

$|W(\mathcal{A}_P)|$ counts in how many ways $2n$ can be written as a sum of three integers that vary in $\{0, 1, \ldots, n-1\}$. Let us denote this integer by $\sigma_n^3(2n)$. It follows, by Appendix B, that $|W(\mathcal{A}_P)| = \sigma_n^3(2n) = (n-2)(n-1)/2$. As for $|W(\mathcal{A})|$, it counts the number of solutions of

$$\frac{1}{n}\sum_{j=1}^{6} a^j = 4 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2 \quad \text{for every } P.$$

This means that $a^1 + a^2 + a^3 = 2n$ and that $a^i = a^j$ if and only if the lines H_i and H_j intersect in a double point of $\mathcal{A}$. Hence $|W(\mathcal{A})| = \sigma_n^3(2n)$. The result follows.

The arrangement dual to the arrangement defined by the inflexion points of a smooth cubic

Another arrangement considered in [8] is the dual to the arrangement defined by the inflexion points of a smooth cubic. Let $\mathcal{A}$ be such an arrangement. It has degree nine and twelve triple points as only singularities. In particular, each line contains four triple points.

Proposition 5.8. *Let $\widetilde{S}_n$ be a desingularization of the standard abelian covering of $\mathbb{P}^2$ branched along $\mathcal{A}$ with $H_\infty \subset \mathcal{A}$ and group $(\mathbb{Z}/n\mathbb{Z})^8$. Then*

$$q(\widetilde{S}_n) = 8(n-1)(n-2) - 2\delta^0_{n \bmod 3},$$

where δ_i^j denotes the Kronecker symbol.

Remark. The case $n = 5$ is treated in [10] and the general case, when n is not divisible by 3, in [15].

Proof. By Theorem 5.6 we have to study edges for which the degree of the corresponding subarrangement $\mathcal{A}_W$ is divisible by 3, *i.e.* equals 3, 6 or 9. In the first case, $\mathcal{A}_W$ is a pencil subarrangement with a single triple point. It will be denoted $\mathcal{A}_P$, with P the triple point. If $\sigma_n^3(2n)$ is, as before, the number of ways $2n$ can be written as a sum of three integers from the set $\{0, 1, \ldots, n-1\}$, then

$$\sum_P |W(\mathcal{A}_P)| \cdot h^1\big(\mathbb{P}^2, \mathcal{I}_P(-1)\big) = 12\,\sigma_n^3(2n).$$

In the second case, $\mathcal{A}_W$ is a Ceva subarrangement and it is easy to see that such a subarrangement cannot exist. As for the last case, there are different edges W such that $\mathcal{A}_W = \mathcal{A}$. Let W be determined by nine points among the twelve triple points—at least nine points are needed so that the corresponding h^1 might be non zero. Since any ten among the twelve points impose independent conditions on cubics as soon as the two remaining points lie on a line of the arrangement, we infer that W is determined by nine points such that there is no line of the arrangement containing any two of the remaining three points. Hence, through each of these three points pass three lines of the arrangement. Now, if Z is the union of the nine points that determine W, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(3)) = 1$. If H_1, H_2, and H_3 are the three lines through one of the triple points not in Z, then summing up the conditions for the points of Z lying on each of these three lines, we obtain that

$$6n = 3a^1 + \sum_{j=4}^{9} a^j = 3a^2 + \sum_{j=4}^{9} a^j = 3a^3 + \sum_{j=4}^{9} a^j.$$

Hence a^j is constant for the lines passing through each of the three missing points. Let $a(W)$, $a'(W)$ and $a''(W)$ be these three values. By the preceding equalities,

$$a(W) + a'(W) + a''(W) = 2n. \tag{20}$$

It follows that

$$\begin{aligned} q(\widetilde{S}_n) &= \sum_P |W(\mathcal{A}_P)| \cdot h^1\big(\mathbb{P}^2, \mathcal{I}_P(-1)\big) \\ &\quad + \sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3)) \\ &\quad + \sum_{\substack{W \text{ given} \\ \text{by 10 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3)) \\ &\quad + |W(\mathcal{A})| \cdot h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})(3)), \end{aligned}$$

since if an edge is defined by eleven points, using (20), it will be defined by all twelve in fact. Moreover, in the two middle sums, $h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3)) = h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3)) = 1$. Indeed, if $U \in \mathcal{U}(W)$ and W is defined by a set Z of nine triple points as above, the subscheme $Z(U)$ is Z. If, let us say, $3a(W) = 2n$, then we obtain a distinguished edge contained in W and defined by 10 points.

From the preceding considerations and since there are exactly four groups[3] of three points such that there is no line of the arrangement containing any two among the three points,

$$\begin{aligned} &\sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3)) + \sum_{\substack{W \text{ given} \\ \text{by 10 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3)) \\ &= \sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \big(|W| - \delta^0_{n \bmod 3} |W(\mathcal{A})|\big) = 4(\sigma^3_n(2n) - \delta^0_{n \bmod 3}). \end{aligned}$$

The corrective term $\delta^0_{n \bmod 3}$ is given by the fact that if n is divisible by 3, then the point in W corresponding to the case $a(W) = a'(W) = a''(W) = 2n/3$ is to be considered in the edge $W(\mathcal{A})$. We conclude that

$$\begin{aligned} q(\widetilde{S}_n) &= 12\sigma^3_n(2n) + 4(\sigma^3_n(2n) - \delta^0_{n \bmod 3}) + 2\delta^0_{n \bmod 3} \\ &= 8(n-1)(n-2) - 2\delta^0_{n \bmod 3}, \end{aligned}$$

since in the hereafter lemma it is shown that $h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})(3)) = 2$. $\square$

Lemma. $h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})) = 2$.

Proof. Let Z denotes the twelve triple points of $\mathcal{A}$. We apply the trace-residual exact sequence to the three lines H_1, H_2 and H_3 that pass through one of the triple points. Let P and P' be the points not lying on these lines. The exact sequences are

$$\begin{aligned} &0 \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_1} Z}(2) \longrightarrow \mathcal{I}_Z(3) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0, \\ &0 \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)}(1) \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_1} Z}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0 \end{aligned}$$

and

$$0 \longrightarrow \mathcal{I}_{P+P'} \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow 0,$$

[3] These four groups are two by two disjoint.

since $\deg \operatorname{Res}_{H_2}(\operatorname{Res}_{H_1} Z) = 3$ and $\operatorname{Res}_{H_3}\big(\operatorname{Res}_{H_2}(\operatorname{Res}_{H_1} Z)\big) = P \cup P'$. It follows that

$$\begin{aligned} h^1(\mathbb{P}^2, \mathcal{I}_Z(3)) &= h^1(\mathbb{P}^2, \mathcal{I}_{\operatorname{Res}_{H_2}(\operatorname{Res}_{H_1} Z)}(1)) \\ &= h^1(\mathbb{P}^2, \mathcal{I}_{P \cup P'}) + h^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 2. \quad \square \end{aligned}$$

The arrangement associated to the Hesse pencil

Let $\mathcal{A}$ be the line arrangement associated to the Hesse pencil. It is composed by the lines of the four singular fibres of the pencil generated by a smooth elliptic curve and its Hessian. It has degree 12 and twelve double points and nine quadruple points as singularities. The double points correspond to intersection points of lines from the same singular fibre.

Proposition 5.9. *Let $\widetilde{S}_n$ be a desingularization of the abelian covering of $\mathbb{P}^2$ branched along $\mathcal{A}$ with $H_\infty \subset \mathcal{A}$ and group $(\mathbb{Z}/n\mathbb{Z})^{11}$. Then*

$$q(\widetilde{S}_n) = \frac{(n-1)(17n^2 + 49n - 146)}{2}.$$

Proof. Theorem 4.2 must be used here. The relevant values for each singular point of $\mathcal{A}$ are 2 and 3. The distinguished edges appearing in the formula for the irregularity are of the following types.

(1) W_P associated to $(P, 2)$ for each singular point P. The corresponding term in the right hand member of (11) equals $\sigma_n^4(2n)$ since the conditions are

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 2 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2.$$

We refer the reader to Appendix B for the definition and the computation of the $\sigma_b^d(N)$ appearing in this subsection.

(2) W_P associated to $(P, 3)$ for each singular point P. Here the corresponding term equals $\sigma_n^4(3n)$.

(3) $W_{\mathcal{B}}$, with $\mathcal{B}$ a Ceva subarrangement. The edge $W_{\mathcal{B}}$ is associated to the singular points of $\mathcal{B}$ seen in $\mathcal{A}$, with relevant value 2 for each one of them. There are 54 such subarrangements, one for each choice of two fibres and two by two components in each fibre—such a choice determines the four triple points of $\mathcal{B}$. As for the terms corresponding to $W_{\mathcal{B}}$ in the formula for the irregularity, let $H_1, \ldots, H_6$ be the lines of $\mathcal{B}$ and let $H_7, \ldots, H_{10}$ be the remaining lines through its triple points. Furthermore we suppose that H_i and H_j intersect in a double point if and only if $i + j = 7$. The defining conditions of $W_{\mathcal{B}}$ are the four equalities corresponding to the triple points:

$$\frac{1}{n}(a^1 + a^2 + a^3 + a^7) = 2 \quad \text{and so on, plus} \quad a^{11} = a^{12} = 0.$$

The corresponding cohomology group in the formula (11) is non trivial if and only if $h_{\mathcal{A}}(W_{\mathcal{B}}) = \sum_{j=1}^{10} a^j/n = 4$. Summing these four conditions for the four points, we get

$$2\sum_{j=1}^{6} a^j + \sum_{k=7}^{10} a^k = 8n.$$

We conclude that $W_{\mathcal{B}}$ must be defined by $a^7 = \cdots = a^{12} = 0$, $a^1 = a^6, a^2 = a^5$ and $a^3 = a^4$, and $a^1 + a^2 + a^3 = 2n$. Hence $|W_{\mathcal{B}}| = \sigma_n^3(2n)$.

(4) W defined by all nine singular points: six with relevant value 2, and the remaining three with relevant value 3. There are three lines H_j that do not pass through the points whose relevant value is 3 and do not intersect in a point. There are 72 such possibilities, $\binom{4}{3} \cdot 3 \cdot 3 \cdot 2$—$\binom{4}{3}$ choices for the fibres with distinguished components, 3 choices for the distinguished component of the first fibre, 3 for the second and 2 for the third. But, applying the trace-residual exact sequence with respect to the three components, we see that h^1 vanishes.

(5) W defined by all nine singular points: six with relevant value 3, and the remaining three with relevant value 2—the configuration obtained from the preceding one by exchanging 2 with 3. As before, $h^1 = 0$ too.

(6) W defined by all nine singular points with relevant value 2. Again h^1 does not vanish if and only if $h_{\mathcal{A}}(W) = 6$. Hence the linear system defining W becomes

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 6 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2 \quad \text{for every singular point } P.$$

Summing up the conditions imposed by the multiple points yields

$$3\sum_{j=1}^{12} a^j = 9 \cdot 2n;$$

hence $\sum_{H_j \ni P} a^j = 2n$ for every P. But then

$$6n = \sum_{P \in H_{j_0}} \sum_{H_j \ni P} a^j = 3a^{j_0} + \sum_{\substack{H_j \text{ not a component of the fibre} \\ \text{that contains } H_{j_0}}} a^j.$$

Hence a^j is constant along each special fibre of the Hesse pencil and $|W| = \sigma_n^4(2n)$.

(7) W defined by all nine singular points with relevant value 3. Arguing as in the previous case, $h_{\mathcal{A}}(W) = 9$ and hence we have

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 9 \quad \text{and} \quad \frac{1}{n}\sum_{j \in \alpha} a^j = 3 \quad \text{for every } \alpha,$$

and eventually $|W| = \sigma_n^4(3n)$. For the computation of the h^1 in this case, the trace residual sequence gives

$$0 \longrightarrow \mathcal{I}_{\sum P}(3) \xrightarrow{u} \mathcal{I}_{\sum 2P}(6) \xrightarrow{r} \mathcal{O}_E \longrightarrow 0,$$

where E is a smooth cubic from the Hesse pencil. Now $h^0(\mathbb{P}^2, \mathcal{I}_{\sum P}(3)) = 2$ and $h^0(\mathbb{P}^2, \mathcal{I}_{\sum 2p_\alpha}(6)) \geq 3$, hence $h^0 r$ is surjective yielding that $h^1(\mathbb{P}^2, \mathcal{I}_{\sum 2p_\alpha}(6)) = 2$.

Summing up and using the computations in Appendix B,

$$\begin{aligned} q(\widetilde{S}_n) &= 9 \cdot \sigma_n^4(2n) + 9 \cdot \sigma_n^4(3n) + 54 \cdot \sigma_n^3(2n) + \sigma_n^4(2n) + 2 \cdot \sigma_n^4(3n) \\ &= 10\frac{(n-1)(2n^2+2n-9)}{3} + 11\frac{(n-1)(n-2)(n-3)}{6} \end{aligned}$$

$$
\begin{aligned}
&+54\frac{(n-1)(n-2)}{2}\\
&=\frac{(n-1)(17n^2+49n-146)}{2}. \quad \square
\end{aligned}
$$

General multiple planes

The last example we would like to consider is the one that makes use of Theorem 4.2 in its full generality. Let $\mathcal{A}$ be the Ceva's arrangement with the lines $C_1, \ldots, C_6$ such that C_i and C_j determine a double point if and only if $i + j = 7$. Let S' be the $(\mathbb{Z}/5\mathbb{Z})^3$-abelian covering of $\mathbb{P}^2$ defined by the reduced building data

$$
\begin{aligned}
&5L_{\chi_1} \sim 3C_2 + C_3 + C_6\\
&5L_{\chi_2} \sim 2C_1 + 2C_2 + C_4\\
&5L_{\chi_3} \sim C_1 + 3C_3 + C_5.
\end{aligned}
$$

It is one of the examples considered by Ishida in [10, Section 6], with $q(S) = 10$, where S is the normalization of S'. In [10] it is shown that this surface is a quotient of the Hirzebruch surface constructed as an $(\mathbb{Z}/5\mathbb{Z})^5$-abelian covering of the plane, by the group $(\mathbb{Z}/5\mathbb{Z})^2$. It also verifies $c_1^2 = c_2$. Moreover it is asserted that the surface is isomorphic to the one constructed by Inoue (see [9]) from the elliptic modular surface of level 5.

Let us show how the irregularity might be computed using Theorem 4.2. There are non-reduced components in the branch locus and C_∞ is taken to be C_6. We have

$$
q(S) = \sum_{P \text{ triple point}} |W(\mathcal{A}_P)|_5^M + |W(\mathcal{A})|_5^M,
$$

where

$$
M = \begin{bmatrix} 0 & 2 & 1\\ 3 & 2 & 0\\ 1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{bmatrix}
$$

and $|W|_5^M = \operatorname{card} \varphi^{-1}(W \cap [0,1)^6)$, with $\varphi : [0,1)^3 \cap (1/5\mathbb{Z})^3 \to [0,1)^6$ defined by

$$
\varphi\left(\frac{a^1}{5}, \frac{a^2}{5}, \frac{a^3}{5}\right) = \left(\left\langle \sum_{j=1}^{3} m_j^1 \frac{a^j}{5} \right\rangle, \ldots, \left\langle \sum_{j=1}^{3} m_j^6 \frac{a^j}{5} \right\rangle\right).
$$

Here as before, $\mathcal{A}_P$ is the pencil subarrangement determined by the triple point P. An easy computation gives $|W(\mathcal{A}_P)|_5^M = 2$ for every triple point. Furthermore, since the equations

$$
\left\langle \frac{2a^2 + a^3}{5} \right\rangle = \left\langle \frac{a^1}{5} \right\rangle, \qquad \left\langle \frac{3a^1 + 2a^2}{5} \right\rangle = \left\langle \frac{a^3}{5} \right\rangle, \qquad \left\langle \frac{a^1 + 3a^3}{5} \right\rangle = \left\langle \frac{a^2}{5} \right\rangle
$$

and $\langle a^1/5 \rangle + \langle a^2/5 \rangle + \langle a^3/5 \rangle = 2$ lead to the only solutions $(2, 4, 4)$ and $(4, 3, 3)$, it follows that $|W(\mathcal{A})|_5^M = 2$ also. Hence the irregularity is 10.

Appendix A. Technical result

In Remark 3.2 (iii) and hence, in the proof of Theorem 4.2, we used two technical results that enabled us to describe the reduced building data of the normalization of a standard covering – which is also a standard covering (see [22, Corollary 3.1]) – in terms of the initial reduced building data. The first result was Proposition 3.2 in [20]. The second is somehow similar and details the fourth step in the normalization algorithm presented in [22], the step peculiar to the abelian situation.

Proposition A.1. *Let X be smooth and let $\pi : Y \to X$ be a standard abelian covering determined by the set of reduced building data $\mathcal{L}_{\chi_j}$ and B_f, $1 \le j \le s$ and $f \in \mathfrak{F}$. Let C be a multiplicity 1 component of both B_f and B_g, i.e. $B_f = C + R_f$ and $B_g = C + R_g$. After the normalization procedure has been applied to C and $Y' \to Y$ is the new surface, if $\chi = \chi_1^{a_1} \cdots \chi_s^{a_s}$, then*

$$\begin{aligned} L'_\chi \sim \sum_{j=1}^{s} a_j L_{\chi_j} &- \left\lfloor \sum_{j=1}^{s} a_j \left(\frac{f(\chi_j)^\bullet}{m_f} + \frac{g(\chi_j)^\bullet}{m_g} \right) \right\rfloor C \\ &- \left\lfloor \sum_{j=1}^{s} \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor R_f - \left\lfloor \sum_{j=1}^{s} \frac{a_j g(\chi_j)^\bullet}{m_g} \right\rfloor R_g \\ &- \sum_{h \ne f,g} \left\lfloor \sum_{j=1}^{s} \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h. \end{aligned}$$

Proof. Assume that $f : \widehat{G} \to \mathbb{Z}/m_f$ and that $g : \widehat{G} \to \mathbb{Z}/m_g$. Let d and m be the greatest common divisor of, and respectively the smallest common multiple of m_f and m_g. If $\varphi : \mathbb{Z}/m_f \times \mathbb{Z}/m_g \to \mathbb{Z}/m$ is defined by $\varphi(1,0) = m_g/d$ and $\varphi(0,1) = m_f/d$, then set $f' : \widehat{G} \to \mathbb{Z}/m_{f'}$ the morphism defined by the composition

$$\widehat{G} \xrightarrow{f \times g} \mathbb{Z}/m_f \times \mathbb{Z}/m_g \xrightarrow{\varphi} \operatorname{Im} \overline{\varphi} \xrightarrow{\iota} \mathbb{Z}/m_{f'}$$

where $\overline{\varphi}$ is the morphism $\varphi \circ (f \times g)$ and ι the isomorphism defined by $\iota(m/m_{f'}) = 1$. The normalization of Y along C is constructed by modifying the covering data as follows:

$$L'_\chi \sim \begin{cases} L_\chi - C, & \text{if } \dfrac{f(\chi)^\bullet}{m_f} + \dfrac{g(\chi)^\bullet}{m_g} \ge 1 \\ L_\chi, & \text{otherwise} \end{cases}$$

and

$$B'_f \sim B_f - C, \qquad B'_g \sim B_g - C, \qquad B'_{f'} \sim B_{f'} + C, \qquad B'_h \sim B_h$$

for $h \ne f, g, f'$.

Applying these modifications to (6) gives

$$\begin{aligned}
L'_\chi &\sim \sum_j a_j L'_{\chi_j} - \sum_{h\in\mathfrak{F}} \left\lfloor \sum_j \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h \\
&\sim {\sum_i}' a_i (L_{\chi_i} - C) + {\sum_k}'' a_k L_{\chi_k} \\
&\quad - \left\lfloor \sum_j \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor R_f - \left\lfloor \sum_j \frac{a_j g(\chi_j)^\bullet}{m_g} \right\rfloor R_g \\
&\quad - \left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor (B_{f'} + C) - \sum_{h\neq f,g,f'} \left\lfloor \sum_j \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h,
\end{aligned}$$

where the sum $\sum'$ runs over those i's for which $f(\chi)^\bullet/m_f + g(\chi)^\bullet/m_g \geq 1$ and $\sum''$ over the other k's. To prove the result it is sufficient to show that

$${\sum_i}' a_i + \left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor = \left\lfloor \sum_{j=1}^{s} a_j \left(\frac{f(\chi_j)^\bullet}{m_f} + \frac{g(\chi_j)^\bullet}{m_g} \right) \right\rfloor.$$

But

$$\begin{aligned}
\left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor &= \left\lfloor \sum_j \frac{a_j}{m_{f'}} \left(\frac{m_g}{d} f(\chi_j)^\bullet + \frac{m_f}{d} g(\chi_j)^\bullet \right)^\bullet \frac{m_{f'}}{m} \right\rfloor \\
&= \left\lfloor {\sum_i}' + {\sum_k}'' \right\rfloor.
\end{aligned}$$

Since for each i in the first sum

$$\left(\frac{m_g}{d} f(\chi_i)^\bullet + \frac{m_f}{d} g(\chi_i)^\bullet \right)^\bullet = \frac{m_g}{d} f(\chi_i)^\bullet + \frac{m_f}{d} g(\chi_i)^\bullet - m$$

and for each k in the second

$$\left(\frac{m_g}{d} f(\chi_k)^\bullet + \frac{m_f}{d} g(\chi_k)^\bullet \right)^\bullet = \frac{m_g}{d} f(\chi_k)^\bullet + \frac{m_f}{d} g(\chi_k)^\bullet,$$

the identity follows. □

Appendix B. Elementary computations

Let d, b, N be positive integers. We denote by $\sigma_b^d(N)$ the number of ways N can be written as an ordered sum of d non negative integers smaller than b. We are mainly interested by $\sigma_n^3(2n)$, $\sigma_n^4(2n)$, and $\sigma_n^4(3n)$.

These numbers are easily computed, for b and N fixed, inductively. We have:

$$\sigma_b^2(N) = \begin{cases} N+1 & \text{if } N < b \\ 2(b-1) - N + 1 & \text{if } b \leq N < 2b-1 \\ 0 & \text{if } 2b-1 \leq N \end{cases}$$

$$\sigma_b^3(N) = \begin{cases} \dfrac{(N+1)(N+2)}{2} & \text{if } N < b \\ \dfrac{b(b+1)}{2} + (2b-2-N)(N-b+1) & \text{if } b \le N < 2b-1 \\ \dfrac{(3b-2-N)(3b-1-N)}{2} & \text{if } 2b-1 \le N < 3b-2 \\ 0 & \text{if } 3b-2 \le N. \end{cases}$$

Thus

$$\sigma_n^3(2n) = \frac{(n-2)(n-1)}{2}$$

and

$$\sigma_n^4(2n) = \frac{(n-1)(2n^2+2n-9)}{3}, \qquad \sigma_n^4(3n) = \frac{(n-3)(n-2)(n-1)}{6}.$$

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